

Representations of $\mathcal{U}_h(\mathfrak{su}(N))$ derived from quantum flag manifolds

Pavel Šťovíček^{*†} and Reidun Twarock

Arnold Sommerfeld Institute
TU Clausthal, Leibnizstr. 10
D-38678 Clausthal-Zellerfeld, Germany

Abstract

A relationship between quantum flag and Grassmann manifolds is revealed. This enables a formal diagonalization of quantum positive matrices. The requirement that this diagonalization defines a homomorphism leads to a left $\mathcal{U}_h(\mathfrak{su}(N))$ – module structure on the algebra generated by quantum antiholomorphic coordinate functions living on the flag manifold. The module is defined by prescribing the action on the unit and then extending it to all polynomials using a quantum version of Leibniz rule. Leibniz rule is shown to be induced by the dressing transformation. For discrete values of parameters occurring in the diagonalization one can extract finite-dimensional irreducible representations of $\mathcal{U}_h(\mathfrak{su}(N))$ as cyclic submodules.

1 INTRODUCTION

Flag manifolds were quantized already some time ago [1, 2]. Also some other types of homogeneous spaces were treated including Grassmann manifolds [3, 4, 5, 6]. Moreover, the quantization can be described in a unified way for all types of coadjoint orbits regarded as complex manifolds and for all simple compact groups from the four principal series [7]. Quantum homogeneous spaces were related to representations and co-representations when taking various points of view like that of induced representations, utilizing q -difference operators etc. [8, 9, 10, 11]. Other aspects are of interest, too, like applications to physical models and differential geometry [12, 13, 14, 15]. In fact, we just succeeded to quote only a small part of contributions related to this subject (cf. Ref. [16]).

In a recent paper [17], carrier spaces of representations of $\mathcal{U}_h(\mathfrak{su}(N))$ were realized as subspaces in the algebra generated by quantum coordinate functions on the

^{*}Humboldt Fellow

[†]On leave of absence from: Department of Mathematics, Faculty of Nuclear Science, CTU, Prague, Czech Republic

flag manifold. The particular case of $SU(3)$ was treated in an analogous way in Ref. [18]. Our goal is to derive a similar description but taking a different approach and presenting some additional results, too. Also the obtained formulas are optically somewhat different though necessarily convertible one into each other. While the method of Ref. [17] is based on a decomposition of the universal R-matrix [19, 20, 21] we start directly from the quantized flag manifold.

Before explaining main features of our approach let us devote a couple of words to the classical case. Quantum flag manifolds can be viewed as quantized orbits of the classical dressing transformation of $SU(N)$ acting on its solvable dual AN [22, 23]. The solvable group AN comes from the Iwasawa decomposition $SL(N, \mathbb{C}) = SU(N) \cdot AN$ and is formed by unimodular upper-triangular matrices having positive diagonals. To make the structure of dressing orbits more transparent one can use the bijection sending $\Lambda \in AN$ to a unimodular positive matrix $M = \Lambda^* \Lambda$. The dressing action on positive matrices becomes just the unitary transformation $M \mapsto U^* M U$ and thus the orbits are determined by sets of eigen-values. We attempted to find a quantum analog to this procedure.

To diagonalize quantum positive matrices we needed, first of all, to reveal a relationship between quantum flag and Grassmann manifolds. It is quite straightforward to see that the quantum Grassmannians are embedded into and jointly generate the quantum algebra related to the flag manifold. Also the constraints reducing the number of generators obtained this way are easy to find. What is less obvious are cross commutation relations between different Grassmannians. The decomposition of the quantum flag manifold into Grassmann manifolds induces a diagonalization of quantum positive matrices provided a set of parameters (eigen-values) has been chosen. The requirement that this diagonalization defines a homomorphism from $\mathcal{A}_q(AN)$ onto the quantum flag manifold leads to a left $\mathcal{U}_h(\mathfrak{su}(N))$ – module structure on the subalgebra of the latter algebra generated by quantum “antiholomorphic” functions. The module is defined by prescribing the action on the unit and then extending it to all polynomials in non-commutative variables (quantum antiholomorphic coordinate functions) using a recursive rule, an idea utilized already in Ref. [24]. Moreover, we prove that this recursive rule follows from the quantum dressing transformation, making the role of the dressing transformation quite explicit. Up to this point, we employed the Faddeev–Reshetikhin–Takhtajan (FRT) description of deformed enveloping algebras [25]. However this result enables us to transcribe, quite straightforwardly, all expressions in terms of Chevalley generators, too. Let us note that in Ref. [17] only the FRT picture has been treated. Finite-dimensional irreducible representations of $\mathcal{U}_h(\mathfrak{su}(N))$ are then easily obtained as cyclic submodules with unit as the cyclic vector and, at the same time, the lowest weight vector.

We have just explained the basic ideas following the structure of the paper. Let us summarize that the notation is introduced and some preliminary facts are reviewed in Section 2, the relationship between quantum flag and Grassmann manifolds is described in Section 3, the left $\mathcal{U}_h(\mathfrak{su}(N))$ – module structure on the quantum flag manifold is derived in Section 4, the role of the quantum dressing transforma-

tion is revealed in Section 5 and Section 6 is devoted to the transcription of all formulas in terms of Chevalley generators as well as to finite-dimensional irreducible representations of $\mathcal{U}_h(\mathfrak{su}(N))$.

2 PRELIMINARIES, NOTATION

Let us recall some basic and well known facts related to the quantum group $SU_q(N)$ and the deformed enveloping algebra $\mathcal{U}_h(\mathfrak{su}(N))$ [26, 27, 28] introducing this way the notation. The deformation parameter is $q = e^{-h}$, with $h \in \mathbb{R}$ (or one can consider, too, h as a formal variable but real like, i.e., $h^* = h$ and to work with the ring $\mathbb{C}[[h]]$).

The $*$ -Hopf algebra of quantum functions living on $SU(N)$ is denoted by $\mathcal{A}_q(SU(N))$ and U stands for the defining $N \times N$ vector corepresentation. The symbols ε and Δ designate everywhere the counit and the comultiplication, respectively, for any Hopf algebra under consideration and we use Sweedler notation (a summation is understood)

$$\Delta Y = Y_{(1)} \otimes Y_{(2)}.$$

Thus

$$\begin{aligned} R_{12}U_1U_2 &= U_2U_1R_{12}, & \det_q U &= 1, & U^* &= U^{-1}, \\ \varepsilon(U) &= \mathbf{I}, & \Delta U &= U \dot{\otimes} U. \end{aligned} \quad (2.1)$$

Here we define, as usual, $(A \dot{\otimes} B)_{jk} := \sum_s A_{js} \otimes B_{sk}$ and $(A^*)_{jk} := (A_{kj})^*$.

The R-matrix acting in $\mathbb{C}^N \otimes \mathbb{C}^N$ and obeying Yang-Baxter (YB) equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ is given by

$$R_{jk,st} := \delta_{js}\delta_{kt} + (q - q^{\text{sgn}(k-j)})\delta_{jt}\delta_{ks}. \quad (2.2)$$

In what follows we identify linear operators in \mathbb{C}^N (and its tensor products) with their matrices in the standard basis $\{e_1, e_2, \dots, e_N\}$. In tensor products the lexicographical ordering is assumed. The R-matrix verifies the relations

$$R_{q^{-1}} = R_q^{-1}, \quad R_{12}^t = R_{21} \equiv PR_{12}P \quad (2.3)$$

and Hecke condition

$$(q - q^{-1})P = R_{12} - R_{21}^{-1} = R_{21} - R_{12}^{-1}. \quad (2.4)$$

Here P , $P_{jk,st} := \delta_{jt}\delta_{ks}$, is the flip operator.

The $*$ -Hopf algebra $\mathcal{U}_h(\mathfrak{su}(N))$ is generated by Chevalley generators $q^{\pm H_j}$, X_j^+ , X_j^- , $1 \leq j \leq N-1$, and is determined by the relations

$$[q^{H_j}, q^{H_k}] = 0,$$

$$\begin{aligned}
q^{H_j} X_k^\pm &= \begin{Bmatrix} q^{\pm 2} \\ q^{\mp 1} \\ 1 \end{Bmatrix} \times X_k^\pm q^{H_j}, \quad \text{for } \begin{cases} j = k, \\ |j - k| = 1, \\ |j - k| \geq 2, \end{cases} \\
[X_j^+, X_k^-] &= \delta_{jk}(q^{H_j} - q^{-H_j})/(q - q^{-1}), \\
(X_j^\pm)^2 X_k^\pm - (q + q^{-1})X_j^\pm X_k^\pm X_j^\pm + X_k^\pm (X_j^\pm)^2 &= 0, \quad \text{for } |j - k| = 1, \\
[X_j^\pm, X_k^\pm] &= 0, \quad \text{for } |j - k| \geq 2,
\end{aligned} \tag{2.5}$$

and

$$(q^{\pm H_j})^* = q^{\pm H_j}, \quad (X_j^\pm)^* = X_j^\mp. \tag{2.6}$$

Furthermore,

$$\begin{aligned}
\varepsilon(q^{\pm H_j}) &= 1, \quad \varepsilon(X_j^\pm) = 0, \\
\Delta(q^{\pm H_j}) &= q^{\pm H_j} \otimes q^{\pm H_j}, \quad \Delta(X_j^\pm) = X_j^\pm \otimes q^{-H_j/2} + q^{H_j/2} \otimes X_j^\pm.
\end{aligned} \tag{2.7}$$

There exists another description of $\mathcal{U}_h(\mathfrak{su}(N))$ due to Faddeev–Reshetikhin–Takhtajan [25] which can be reinterpreted as the quantization of the generalized dual of $SU(N)$, namely the solvable group AN coming from the Iwasawa decomposition $SL(N, \mathbb{C}) = SU(N) \cdot AN$. The $*$ -Hopf algebra $\mathcal{A}_q(AN)$ is generated by entries of the upper triangular matrix $\Lambda = (\alpha_{jk})$ and its adjoint Λ^* and is determined by the relations

$$\begin{aligned}
R_{12}\Lambda_1\Lambda_2 &= \Lambda_2\Lambda_1R_{12}, \quad \Lambda_1^*R_{12}^{-1}\Lambda_2 = \Lambda_2R_{12}^{-1}\Lambda_1^*, \\
\alpha_{jj}^* &= \alpha_{jj}, \quad \prod_{j=1}^N \alpha_{jj} = 1, \\
\varepsilon(\Lambda) &= \mathbf{I}, \quad \Delta_{AN}(\Lambda) = \Lambda \dot{\otimes} \Lambda,
\end{aligned} \tag{2.8}$$

The $*$ -algebras $\mathcal{U}_h(\mathfrak{su}(N))$ and $\mathcal{A}_q(AN)$ can be identified by an isomorphism which is given explicitly in terms of generators ($1 \leq j \leq N - 1$),

$$\begin{aligned}
q^{H_j} &= \alpha_{jj}^{-1} \alpha_{j+1,j+1}, \\
(q - q^{-1})X_j^+ &= -q^{-1/2}(\alpha_{jj}\alpha_{j+1,j+1})^{-1/2} \alpha_{j,j+1}^*, \\
(q - q^{-1})X_j^- &= -q^{-1/2}(\alpha_{jj}\alpha_{j+1,j+1})^{-1/2} \alpha_{j,j+1},
\end{aligned} \tag{2.9}$$

We note that the diagonal elements α_{jj} mutually commute and $\alpha_{jj}\alpha_{j+1,j+1}$ commutes with $\alpha_{j,j+1}$. However $\mathcal{U}_h(\mathfrak{su}(N))$ and $\mathcal{A}_q(AN)$ are opposite as coalgebras. Thus one has to be careful about the comultiplication and this is why we emphasized its origin in (2.8). To avoid any ambiguity, throughout the text Δ is always assumed to come from $\mathcal{U}_h(\mathfrak{su}(N))$ rather than from $\mathcal{A}_q(AN)$.

It turns out as more convenient to work with the $*$ -algebra $\tilde{\mathcal{A}}_d \subset \mathcal{A}_q(AN)$ generated by entries of the matrix $M := \Lambda^* \Lambda$ rather than directly with $\mathcal{A}_q(AN)$. It is straightforward to see that $\tilde{\mathcal{A}}_d$ is determined by the relations

$$M_2 R_{12}^{-1} M_1 R_{21}^{-1} = R_{12}^{-1} M_1 R_{21}^{-1} M_2, \quad M^* = M. \tag{2.10}$$

In fact, one is not loosing that much as the algebra $\mathcal{A}_q(AN)$ can be, in principle, again recovered from $\tilde{\mathcal{A}}_d$ when decomposing M into a product of lower and upper triangular matrices (cf. Proposition 3.2 in Ref. [6]).

An important feature is the duality between $\mathcal{U}_h(\mathfrak{su}(N))$ and $\mathcal{A}_q(SU(N))$ expressed in terms of a non-degenerate pairing. The both structures are combined according to the rules

$$\langle X, fg \rangle = \langle X_{(1)}, f \rangle \langle X_{(2)}, g \rangle, \quad \langle XY, f \rangle = \langle X, f_{(1)} \rangle \langle Y, f_{(2)} \rangle, \quad (2.11)$$

with $\Delta X = X_{(1)} \otimes X_{(2)}$, $\Delta f = f_{(1)} \otimes f_{(2)}$, and

$$\langle X, 1 \rangle = \varepsilon(X), \quad \langle 1, f \rangle = \varepsilon(f). \quad (2.12)$$

The pairing can be described explicitly in terms of generators. Let E_{jk} be the matrix units acting as rank-one operators: $E_{jk}v := (e_k, v)e_j$ (the indices shouldn't be confused with the leg notation referring to tensor products). Then we have

$$\langle q^{\pm H_j}, U \rangle = q^{\pm(E_{jj} - E_{j+1, j+1})}, \quad \langle X_j^+, U \rangle = E_{j, j+1}, \quad \langle X_j^-, U \rangle = E_{j+1, j}. \quad (2.13)$$

In the FRT picture we have

$$\langle \Lambda_1, U_2 \rangle = R_{21}^{-1}, \quad \langle \Lambda_1^*, U_2 \rangle = R_{12}^{-1}. \quad (2.14)$$

To describe a relationship between quantum flag and Grassmann manifolds we shall need the following family of orthogonal projectors. Let $E^{(m)}$ be the matrix corresponding to the orthogonal projector in \mathbb{C}^N onto $\text{span}\{e_1, \dots, e_m\}$ and set $F^{(m)} := \mathbf{I} - E^{(m)}$. Thus

$$E^{(m)} = \sum_{j \leq m} E_{jj}, \quad F^{(m)} = \sum_{j > m} E_{jj};$$

particularly, $E^{(0)} = \mathbf{0}$, $E^{(N)} = \mathbf{I}$. Quite important is the following relation between $E^{(m)}$ and the R-matrix,

$$E_1^{(m)} R_{12} = E_1^{(m)} R_{12} E_1^{(m)}, \quad (2.15a)$$

and consequently,

$$R_{12} E_2^{(m)} = E_2^{(m)} R_{12} E_2^{(m)}, \quad R_{12} F_1^{(m)} = F_1^{(m)} R_{12} F_1^{(m)}, \quad F_2^{(m)} R_{12} = F_2^{(m)} R_{12} F_2^{(m)}. \quad (2.15b)$$

Observe also that

$$E_1^{(m)} F_2^{(m)} R_{12} = E_1^{(m)} F_2^{(m)}, \quad (2.16)$$

and

$$E_1^{(m)} E_2^{(m)} R_{12} = R_{12} E_1^{(m)} E_2^{(m)}, \quad F_1^{(m)} F_2^{(m)} R_{12} = R_{12} F_1^{(m)} F_2^{(m)}. \quad (2.17)$$

3 QUANTUM FLAG AND GRASSMANN MANIFOLDS

There is a standard way of introducing local holomorphic coordinate functions on homogeneous spaces $SU(N)/S(U(m_1) \times \dots \times U(m_k))$, $\sum m_j = N$, which is given by Gauss decomposition. This coordinate system is well defined on the so called big cell (the unique cell of the top dimension in the cell decomposition) which covers the whole manifold up to an algebraic subset. The coordinate functions appear as entries of a block upper-triangular matrix \mathcal{N} with unit blocks on the diagonal. The structure of the blocks depends on the type of the homogeneous space. The quantization procedure for the algebra of (anti)holomorphic functions living on the big cell has been performed successfully in many particular cases [1, 2, 9, 6]. But there is a unified and compact way of writing down the commutation relations which is valid for any homogeneous space of the above type, namely [7]

$$R_{12}Q_{12}^{-1}\mathcal{N}_1Q_{12}\mathcal{N}_2 = Q_{21}^{-1}\mathcal{N}_2Q_{21}\mathcal{N}_1R_{12}, \quad (3.1)$$

where the matrix Q is obtained from R by annullating some entries in dependence on the concrete homogeneous space in question. Let us specialize (3.1) to flag and Grassmann manifolds.

In the case of the flag manifold, \mathcal{N} is simply an upper triangular matrix with units on the diagonal and we redenote it as \mathcal{Z} ,

$$\mathcal{Z} = (\zeta_{jk}), \quad 1 \leq j, k \leq N, \quad \text{with } \zeta_{jj} = 1, \quad \zeta_{jk} = 0 \text{ for } j > k, \quad (3.2)$$

and $Q = \text{diag}(R)$, $\text{diag}(R)_{jk,st} = q^{\delta_{jk}}\delta_{js}\delta_{kt}$. As $\text{diag}(R)$ commutes with R it is possible to simplify (3.1),

$$R_{12}\mathcal{Z}_1\text{diag}(R)\mathcal{Z}_2 = \mathcal{Z}_2\text{diag}(R)\mathcal{Z}_1R_{12}. \quad (3.3)$$

One can rewrite (3.3) in terms of matrix entries,

$$q^{\delta_{ks}}\zeta_{js}\zeta_{kt} - q^{\delta_{jt}}\zeta_{kt}\zeta_{js} = (q^{\text{sgn}(k-j)} - q^{\text{sgn}(s-t)})q^{\delta_{js}}\zeta_{ks}\zeta_{jt}. \quad (3.4)$$

The relations (3.3) (or (3.4)) define an algebra of quantum holomorphic functions generated by ζ_{jk} , $1 \leq j < k \leq N$, and denoted here by \mathcal{F}_{hol} while the adjoint relations define an algebra of quantum antiholomorphic functions generated by ζ_{jk}^* , $1 \leq j < k \leq N$, and denoted by \mathcal{F}_{ahol} .

In the case of the Grassmann manifold formed by m -dimensional subspaces in \mathbb{C}^N , $1 \leq m \leq N-1$, \mathcal{N} has the block structure given by

$$\mathcal{N} = \begin{pmatrix} \mathbf{I} & Z^{(m)} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \quad \text{with } Z^{(m)} = (z_{jk}^{(m)}), \quad 1 \leq j \leq m, \quad m+1 \leq k \leq N. \quad (3.5)$$

Thus the dimension of the block $Z^{(m)}$ is $m \times (N-m)$. Here and everywhere in what follows we don't specify the dimensions of zero and unit blocks for all cases are determined implicitly in an unambiguous way. We set also

$$\mathfrak{Z}^{(m)} := \begin{pmatrix} \mathbf{0} & Z^{(m)} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad (3.6)$$

so that $\mathcal{N} = \mathbf{I} + \mathfrak{Z}^{(m)}$ and $\mathfrak{Z}^{(m)} = E^{(m)} \mathfrak{Z}^{(m)} F^{(m)}$. Whenever convenient we shall define both $\mathfrak{Z}^{(0)}$ and $\mathfrak{Z}^{(N)}$ as zero matrices. The matrix Q is now related to R according to (cf. (2.15))

$$Q_{12} := E_2^{(m)} R_{12} E_2^{(m)} + F_2^{(m)} R_{12} F_2^{(m)} = R_{12} E_2^{(m)} + F_2^{(m)} R_{12}. \quad (3.7a)$$

It holds also true that

$$Q_{12} = E_1^{(m)} R_{12} E_1^{(m)} + F_1^{(m)} R_{12} F_1^{(m)} = E_1^{(m)} R_{12} + R_{12} F_1^{(m)} \quad (3.7b)$$

and $Q_q^{-1} = Q_{q^{-1}}$. To simplify (3.1) it suffices to multiply this relation by $E_1^{(m)} E_2^{(m)}$ from the left, to use (2.15), (2.17), (3.7) and to observe that $E^{(m)} \mathcal{N} = E^{(m)} + \mathfrak{Z}^{(m)}$. The result is

$$R_{21}(E_1^{(m)} + \mathfrak{Z}_1^{(m)}) R_{12}(E_2^{(m)} + \mathfrak{Z}_2^{(m)}) = (E_2^{(m)} + \mathfrak{Z}_2^{(m)}) R_{21}(E_1^{(m)} + \mathfrak{Z}_1^{(m)}) R_{12}. \quad (3.8)$$

In fact, this relation is equivalent to [6]

$$R_{21}^{[m]} Z_1^{(m)} Z_2^{(m)} = Z_2^{(m)} Z_1^{(m)} R_{12}^{[N-m]}, \quad (3.9)$$

where $R^{[m]}$ stands for the $m^2 \times m^2$ R-matrix related to the quantum group $SU_q(m)$ and $R^{[1]} := q$. One can rewrite (3.8) in terms of entries $z_{jk}^{(m)}$,

$$z_{jk}^{(m)} z_{st}^{(m)} - z_{st}^{(m)} z_{jk}^{(m)} = (q^{\text{sgn}(j-s)} - q^{\text{sgn}(k-t)}) z_{sk}^{(m)} z_{jt}^{(m)}. \quad (3.10)$$

For a given m , $1 \leq m \leq N-1$, the relations (3.8) (or (3.9) or (3.10)) define an algebra of quantum holomorphic functions generated by $z_{jk}^{(m)}$, $1 \leq j \leq m < k \leq N$, and denoted by $\mathcal{G}_{hol}^{(m)}$ while the adjoint relations define an algebra of quantum antiholomorphic functions generated by $z_{jk}^{(m)*}$, $1 \leq j \leq m < k \leq N$, and denoted by $\mathcal{G}_{ahol}^{(m)}$.

The main goal of this section is to express \mathcal{F}_{hol} in terms of $\mathcal{G}_{hol}^{(m)}$, $1 \leq m \leq N-1$ (and analogously for the antiholomorphic versions). It is quite straightforward to embed the algebras $\mathcal{G}_{hol}^{(m)}$ into \mathcal{F}_{hol} as well as to find the constraining relations (the algebras $\mathcal{G}_{hol}^{(m)}$ are not mutually independent as subalgebras in \mathcal{F}_{hol}). A more difficult problem is to determine the cross commutation relations between $\mathcal{G}_{hol}^{(m)}$ and $\mathcal{G}_{hol}^{(n)}$ for $m \neq n$. We introduce some additional notation used only locally in this section and only for the sake of derivation of these relations. Set

$$\mathcal{X}^{(m)} := \mathcal{Z} E^{(m)} = E^{(m)} \mathcal{Z} E^{(m)}, \quad \bar{\mathcal{X}}^{(m)} := \mathcal{Z}^{-1} E^{(m)} = E^{(m)} \mathcal{Z}^{-1} E^{(m)}, \quad \mathcal{Y}^{(m)} := E^{(m)} \mathcal{Z} F^{(m)}. \quad (3.11)$$

Thus we have $E^{(m)} \mathcal{Z} = \mathcal{X}^{(m)} + \mathcal{Y}^{(m)}$ and

$$\mathcal{Z}^{-1} E^{(m)} \mathcal{Z} = \bar{\mathcal{X}}^{(m)} (\mathcal{X}^{(m)} + \mathcal{Y}^{(m)}), \quad E^{(m)} = \bar{\mathcal{X}}^{(m)} \mathcal{X}^{(m)} = \mathcal{X}^{(m)} \bar{\mathcal{X}}^{(m)}. \quad (3.12)$$

Next we will rewrite the commutation relation (3.3) in terms of $\mathcal{X}^{(m)}$ and $\mathcal{Y}^{(m)}$.

Lemma 3.1. Assume that $0 \leq m \leq n \leq N$. It holds true that

$$R_{21}^{-1} \mathcal{X}_1^{(m)} \text{diag}(R) \mathcal{X}_2^{(n)} = \mathcal{X}_2^{(n)} \text{diag}(R) \mathcal{X}_1^{(m)} R_{21}^{-1} E_1^{(m)}, \quad (3.13)$$

$$R_{21}^{-1} \mathcal{X}_1^{(m)} \text{diag}(R) \mathcal{Y}_2^{(n)} = \mathcal{Y}_2^{(n)} \mathcal{X}_1^{(m)}, \quad (3.14)$$

$$E_1^{(m)} R_{12} \mathcal{Y}_1^{(m)} \text{diag}(R) \mathcal{X}_2^{(n)} = \mathcal{X}_2^{(n)} \text{diag}(R) \mathcal{Y}_1^{(m)} R_{12} F_1^{(m)} E_2^{(n)}, \quad (3.15)$$

$$E_1^{(m)} R_{12} \mathcal{Y}_1^{(m)} \text{diag}(R) \mathcal{Y}_2^{(n)} = \mathcal{Y}_2^{(n)} \mathcal{Y}_1^{(m)} R_{12} + (q - q^{-1}) \mathcal{X}_2^{(n)} F_2^{(m)} \mathcal{Y}_1^{(m)} F_1^{(n)} P. \quad (3.16)$$

Proof. To derive (3.13) – (3.16) it suffices to multiply the equation (3.3) by $E_1^{(m)} E_2^{(n)}$ from the left in all cases and by $E_1^{(m)} E_2^{(n)}$, $E_1^{(m)} F_2^{(n)}$, $F_1^{(m)} E_2^{(n)}$, $F_1^{(m)} F_2^{(n)}$, respectively, from the right. One has to employ (2.15), (2.16), (2.17) and, where convenient, (2.4) to commute the projectors with the R-matrix getting consequently, for example,

$$E_1^{(m)} E_2^{(n)} R_{12} = E_1^{(m)} E_1^{(n)} E_2^{(n)} R_{12} = E_1^{(m)} R_{12} E_1^{(m)} E_2^{(n)}$$

(in all cases) and similar relations like

$$R_{12} E_1^{(m)} F_2^{(n)} = E_1^{(m)} F_2^{(n)} + (q - q^{-1}) P E_1^{(m)} F_2^{(n)}$$

(in the case of (3.14)). Note also that $\mathcal{X}^{(m)} = \mathcal{X}^{(n)} E^{(m)}$ etc. We omit further details. ■

We shall need also the following lemma whose verification is quite easy.

Lemma 3.2. It holds true that

$$E_1^{(n)} R_{12}^{\pm 1} F_1^{(n-1)} = \text{diag}(R)^{\pm 1} E_1^{(n)} F_1^{(n-1)}, \quad F_2^{(n-1)} R_{12}^{\pm 1} E_2^{(n)} = \text{diag}(R)^{\pm 1} E_2^{(n)} F_2^{(n-1)}, \quad (3.17)$$

and consequently,

$$m \geq n \implies R_{12}^{\pm 1} E_1^{(m)} F_1^{(m-1)} E_2^{(n)} F_2^{(n-1)} = \text{diag}(R)^{\pm 1} E_1^{(m)} F_1^{(m-1)} E_2^{(n)} F_2^{(n-1)}. \quad (3.18)$$

Evidently,

$$m \neq n \implies \text{diag}(R)^{\pm 1} E_1^{(m)} F_1^{(m-1)} E_2^{(n)} F_2^{(n-1)} = E_1^{(m)} F_1^{(m-1)} E_2^{(n)} F_2^{(n-1)}. \quad (3.19)$$

Proposition 3.3. The algebras $\mathcal{G}_{hol}^{(m)}$, $1 \leq m \leq N - 1$, are embedded into \mathcal{F}_{hol} via the equalities (abusing notation, the LHS should be understood as the result of the embedding)

$$E^{(m)} + \mathfrak{Z}^{(m)} = \mathcal{Z}^{-1} E^{(m)} \mathcal{Z}. \quad (3.20)$$

Moreover, being embedded into \mathcal{F}_{hol} , the subalgebras $\mathcal{G}_{hol}^{(m)}$ generate jointly \mathcal{F}_{hol} for one can express

$$\mathcal{Z} = \sum_{m=1}^N F^{(m-1)} (E^{(m)} + \mathfrak{Z}^{(m)}) = \mathbf{I} + \sum_{m=1}^{N-1} F^{(m-1)} \mathfrak{Z}^{(m)}. \quad (3.21)$$

Thus, on the other hand, \mathcal{F}_{hol} is isomorphic to the algebra generated by the entries of the blocks $Z^{(m)}$, $1 \leq m \leq N - 1$, and determined by the relations

$$1 \leq m < n \leq N - 1 \implies (E^{(m)} + \mathfrak{Z}^{(m)}) (E^{(n)} + \mathfrak{Z}^{(n)}) = E^{(m)} + \mathfrak{Z}^{(m)}, \quad (3.22)$$

$$1 \leq m \leq n \leq N-1 \implies R_{21}(E_1^{(m)} + \mathfrak{Z}_1^{(m)})R_{12}(E_2^{(n)} + \mathfrak{Z}_2^{(n)}) = (E_2^{(n)} + \mathfrak{Z}_2^{(n)})R_{21}(E_1^{(m)} + \mathfrak{Z}_1^{(m)})R_{12}. \quad (3.23)$$

Remarks. (1) The equality (3.20) makes sense also for $m = 0$ and $m = N$ when it reduces to the trivial identities $\mathbf{0} = \mathbf{0}$ and $\mathbf{I} = \mathbf{I}$, respectively. A similar remark applies also for other equalities.

(2) Note that

$$\mathcal{Z}^{-1}E^{(m)}\mathcal{Z} = E^{(m)}\mathcal{Z}^{-1}E^{(m)}\mathcal{Z} \quad \text{and} \quad \mathcal{Z}^{-1}E^{(m)}\mathcal{Z}E^{(m)} = \mathcal{Z}^{-1}\mathcal{Z}E^{(m)} = E^{(m)}$$

and thus the RHS of (3.20) has the same structure of blocks as the LHS. Furthermore, the inverted relation (3.21) follows immediately from the fact that $\text{diag}(\mathcal{Z}) = \mathbf{I}$ and so

$$F^{(m-1)}\mathcal{Z}^{-1}E^{(m)} = F^{(m-1)}E^{(m)} = E^{(m)} - E^{(m-1)},$$

which implies

$$F^{(m-1)}(E^{(m)} + \mathfrak{Z}^{(m)}) = (E^{(m)} - E^{(m-1)})\mathcal{Z}.$$

(3) (3.22) can be rewritten in a way coinciding formally with the classical constraint,

$$m < n \implies \begin{pmatrix} \mathbf{I} & Z^{(m)} \end{pmatrix} \begin{pmatrix} -Z^{(n)} \\ \mathbf{I} \end{pmatrix} = 0, \quad (3.24)$$

and, in fact, it reduces the number of generators. Observe also that it holds trivially true, just by the structure of blocks (cf. (3.6)), that

$$m \leq n \implies (E^{(n)} + \mathfrak{Z}^{(n)})(E^{(m)} + \mathfrak{Z}^{(m)}) = E^{(m)} + \mathfrak{Z}^{(m)}. \quad (3.25)$$

(4) (3.23) for $m = n$ reproduces the defining relation (3.8) of $\mathcal{G}_{hol}^{(m)}$ and for $m < n$ gives the desired cross commutation relations between different quantum Grassmannians.

Proof. We assume that $m \leq n$ and write, for simplicity of notation, \mathcal{X} instead of $\mathcal{X}^{(m)}$, \mathcal{X}' instead of $\mathcal{X}^{(n)}$ and similarly for \mathcal{Y} , \mathcal{Y}' , E , E' etc. Thus we have, for example, $EE' = E'E = E$.

Let us first show that the matrices $\mathfrak{Z}^{(m)}$ defined by the RHS of (3.20) verify (3.22). Since $\mathcal{X} + \mathcal{Y} = E(\mathcal{X}' + \mathcal{Y}')$ and $(\tilde{\mathcal{X}}(\mathcal{X} + \mathcal{Y}))^2 = \tilde{\mathcal{X}}(\mathcal{X} + \mathcal{Y})$ we have

$$\begin{aligned} (E + \mathfrak{Z})(E' + \mathfrak{Z}') &= \tilde{\mathcal{X}}(\mathcal{X} + \mathcal{Y})\tilde{\mathcal{X}}'(\mathcal{X}' + \mathcal{Y}') = \tilde{\mathcal{X}}E\mathcal{X}'\tilde{\mathcal{X}}'(\mathcal{X}' + \mathcal{Y}')\tilde{\mathcal{X}}'(\mathcal{X}' + \mathcal{Y}') \\ &= \tilde{\mathcal{X}}(\mathcal{X} + \mathcal{Y}) = E + \mathfrak{Z}. \end{aligned}$$

Next we will show that the same matrices $\mathfrak{Z}^{(m)}$ verify also (3.23). One can derive successively

$$\begin{aligned} \text{diag}(R)\mathcal{Y}_1R_{12}F_1\tilde{\mathcal{X}}_2'\mathcal{Y}_2' &= \tilde{\mathcal{X}}_2'E_1R_{12}\mathcal{Y}_1\text{diag}(R)\mathcal{Y}_2' && \text{, by (3.15),} \\ &= \tilde{\mathcal{X}}_2'\mathcal{Y}_2'\mathcal{Y}_1R_{12} + (q - q^{-1})E_2'F_2\mathcal{Y}_1F_1'P && \text{, by (3.16),} \end{aligned}$$

whence $(\text{diag}(R)^{-1}F_2E_1 = F_2E_1)$

$$\bar{\mathcal{X}}_1\mathcal{Y}_1R_{12}F_1\bar{\mathcal{X}}'_2\mathcal{Y}'_2 = \bar{\mathcal{X}}_1\text{diag}(R)^{-1}\bar{\mathcal{X}}'_2\mathcal{Y}'_2\mathcal{Y}_1R_{12} + (q - q^{-1})\bar{\mathcal{X}}_1\mathcal{Y}_1F'_1E'_2F_2P. \quad (3.26)$$

Furthermore,

$$\begin{aligned} \bar{\mathcal{X}}_1\text{diag}(R)^{-1}\bar{\mathcal{X}}'_2\mathcal{Y}'_2\mathcal{X}_1 &= \bar{\mathcal{X}}_1\text{diag}(R)^{-1}\bar{\mathcal{X}}'_2R_{21}^{-1}\mathcal{X}_1\text{diag}(R)\mathcal{Y}'_2, \quad \text{by (3.14),} \\ &= \bar{\mathcal{X}}_1\text{diag}(R)^{-1}E'_2\text{diag}(R)\mathcal{X}_1R_{21}^{-1}E_1\bar{\mathcal{X}}'_2\mathcal{Y}'_2, \quad \text{by (3.13),} \\ &= E_1E'_2R_{21}^{-1}E_1\bar{\mathcal{X}}'_2\mathcal{Y}'_2 \\ &= R_{21}^{-1}E_1\bar{\mathcal{X}}'_2\mathcal{Y}'_2, \end{aligned}$$

whence

$$\bar{\mathcal{X}}_1\text{diag}(R)^{-1}\bar{\mathcal{X}}'_2\mathcal{Y}'_2 = R_{21}^{-1}\bar{\mathcal{X}}'_2\mathcal{Y}'_2\bar{\mathcal{X}}_1. \quad (3.27)$$

Combination of (3.26) and (3.27) yields $(\mathfrak{Z} = \bar{\mathcal{X}}\mathcal{Y})$

$$\mathfrak{Z}_1R_{12}F_1\mathfrak{Z}'_2 = R_{21}^{-1}\mathfrak{Z}'_2\mathfrak{Z}_1R_{12} + (q - q^{-1})\mathfrak{Z}_1F'_1E'_2F_2P.$$

This relation can be rewritten as

$$R_{21}\mathfrak{Z}_1R_{12}\mathfrak{Z}'_2 = \mathfrak{Z}'_2R_{21}\mathfrak{Z}_1R_{12} + (q - q^{-1})R_{21}\mathfrak{Z}_1E_2\mathfrak{Z}'_1P + (q - q^{-1})R_{21}\mathfrak{Z}_1F'_1E'_2F_2P.$$

To see this it suffices to notice (for $\mathfrak{Z} = E\mathfrak{Z}F$) that

$$F'_2R_{21}E_1 = E_1F'_2, \quad F_1R_{12}F_1 = F_1R_{12} - (q - q^{-1})F_1E_2P.$$

Next observe that (3.22) (already proven) means

$$\mathfrak{Z}\mathfrak{Z}' = \mathfrak{Z}F' - E\mathfrak{Z}'.$$

Thus we arrive at $(E = EE')$

$$R_{21}\mathfrak{Z}_1R_{12}\mathfrak{Z}'_2 - \mathfrak{Z}'_2R_{21}\mathfrak{Z}_1R_{12} = (q - q^{-1})(R_{21}\mathfrak{Z}_1F'_1E'_2P - R_{21}E_1\mathfrak{Z}'_1E_2P). \quad (3.28)$$

To pass from (3.28) to (3.23) one can use that

$$R_{21}E_1R_{12}E'_2 = E'_2R_{21}E_1R_{12}E'_2 = E'_2R_{21}E_1R_{12}$$

as well as

$$\begin{aligned} \mathfrak{Z}'_2R_{21}E_1R_{12} - R_{21}E_1R_{12}\mathfrak{Z}'_2 &= -(q - q^{-1})R_{21}E_1\mathfrak{Z}'_1E_2P, \\ E'_2R_{21}\mathfrak{Z}_1R_{12} - R_{21}\mathfrak{Z}_1R_{12}E'_2 &= (q - q^{-1})R_{21}\mathfrak{Z}_1F'_1E'_2P. \end{aligned}$$

The last two relations follow respectively from

$$R_{21}E_1 = E_1R_{12}^{-1} + (q - q^{-1})E_1E_2P, \quad E'_2R_{21}E_1 = R_{21}E'_2E_1.$$

To complete the proof we have to consider, on the contrary, the algebra generated by the entries of blocks $Z^{(m)}$, $1 \leq m \leq N-1$, and determined by the relations (3.22), (3.23) and to interpret the equality (3.21) as the defining relation for \mathcal{Z} . Clearly, \mathcal{Z} defined this way is upper triangular with units on the diagonal and

$$E^{(m)}\mathcal{Z} = \sum_{n=1}^m F^{(n-1)}(E^{(n)} + \mathfrak{Z}^{(n)}) = \mathcal{Z}(E^{(m)} + \mathfrak{Z}^{(m)}),$$

by (3.22) and (3.25) ($F^{(n-1)}E^{(m)} = 0$ for $n \geq m+1$). Thus it remains to show that this matrix \mathcal{Z} verifies also (3.3). Using (3.17) one derives

$$\begin{aligned} R_{12}\mathcal{Z}_1\text{diag}(R)\mathcal{Z}_2 &= \sum_{m=1}^N \sum_{n=1}^N R_{12}F_1^{(m-1)}(E_1^{(m)} + \mathfrak{Z}_1^{(m)})\text{diag}(R)F_2^{(n-1)}(E_2^{(n)} + \mathfrak{Z}_2^{(n)}) \\ &= \sum_{m=1}^N \sum_{n=1}^N R_{12}F_1^{(m-1)}F_2^{(n-1)}(E_1^{(m)} + \mathfrak{Z}_1^{(m)})R_{12}(E_2^{(n)} + \mathfrak{Z}_2^{(n)}). \end{aligned}$$

Split the last double sum into two pieces according to whether $m \leq n$ or $m > n$. Using (3.23) or its equivalent

$$m \geq n \implies R_{12}^{-1}(E_1^{(m)} + \mathfrak{Z}_1^{(m)})R_{12}(E_2^{(n)} + \mathfrak{Z}_2^{(n)}) = (E_2^{(n)} + \mathfrak{Z}_2^{(n)})R_{21}(E_1^{(m)} + \mathfrak{Z}_1^{(m)})R_{21}^{-1}$$

one finds that this double sum equals

$$\begin{aligned} &\sum_{m \leq n} R_{12}F_1^{(m-1)}F_2^{(n-1)}R_{21}^{-1}(E_2^{(n)} + \mathfrak{Z}_2^{(n)})R_{21}(E_1^{(m)} + \mathfrak{Z}_1^{(m)})R_{12} \\ &\quad + \sum_{m > n} R_{12}F_1^{(m-1)}F_2^{(n-1)}R_{12}(E_2^{(n)} + \mathfrak{Z}_2^{(n)})R_{21}(E_1^{(m)} + \mathfrak{Z}_1^{(m)})R_{21}^{-1} \\ &= \sum_m F_2^{(m-1)}(E_2^{(m)} + \mathfrak{Z}_2^{(m)})\text{diag}(R)F_1^{(m-1)}(E_1^{(m)} + \mathfrak{Z}_1^{(m)})R_{12} \\ &\quad + \sum_{m < n} (R_{21}^{-1} + (q - q^{-1})P)F_2^{(n-1)}(E_2^{(n)} + \mathfrak{Z}_2^{(n)})\text{diag}(R)F_1^{(m-1)}(E_1^{(m)} + \mathfrak{Z}_1^{(m)})R_{12} \\ &\quad + \sum_{m > n} F_2^{(n-1)}(E_2^{(n)} + \mathfrak{Z}_2^{(n)})\text{diag}(R)F_1^{(m-1)}(E_1^{(m)} + \mathfrak{Z}_1^{(m)})(R_{12} - (q - q^{-1})P) \\ &= \mathcal{Z}_2\text{diag}(R)\mathcal{Z}_1R_{12}. \end{aligned}$$

Here we have used repeatedly (3.17), (3.18), (3.19); for example,

$$\begin{aligned} m < n &\implies PF_2^{(n-1)}(E_2^{(n)} + \mathfrak{Z}_2^{(n)})\text{diag}(R)F_1^{(m-1)}(E_1^{(m)} + \mathfrak{Z}_1^{(m)})R_{12} \\ &= F_1^{(n-1)}F_2^{(m-1)}(E_1^{(n)} + \mathfrak{Z}_1^{(n)})R_{12}(E_2^{(m)} + \mathfrak{Z}_2^{(m)})R_{21}P \\ &= F_1^{(n-1)}F_2^{(m-1)}(E_2^{(m)} + \mathfrak{Z}_2^{(m)})\text{diag}(R)(E_1^{(n)} + \mathfrak{Z}_1^{(n)})P, \end{aligned}$$

and

$$m < n \implies R_{21}^{-1}F_1^{(m-1)}E_1^{(m)}F_2^{(n-1)}E_2^{(n)} = F_1^{(m-1)}E_1^{(m)}F_2^{(n-1)}E_2^{(n)}. \quad \blacksquare$$

4 A CONSTRUCTION OF LEFT $\mathcal{U}_h(\mathfrak{su}(N))$ – MODULES

The goal of this section is to equip the algebra \mathcal{F}_{ahol} with the structure of a left $\mathcal{U}_h(\mathfrak{su}(N))$ – module depending on $(N - 1)$ parameters. More precisely, here we will use the FRT description and hence deal with the algebra $\tilde{\mathcal{A}}_d$ (more or less equivalent to $\mathcal{A}_q(AN)$) introduced in Section 2 (cf. (2.10)). The module will be redefined in terms of Chevalley generators in Section 6. Moreover, the number of parameters is N for the $\tilde{\mathcal{A}}_d$ –module but the expressions in Chevalley generators will turn out to depend only on their ratios and so the number is reduced by one (as it should be).

Assume for a moment that we are able to quantize the big cell of the flag manifold with its real analytic structure, that means to construct a $*$ -algebra \mathcal{F} generated jointly by ζ_{jk} and ζ_{st}^* ($1 \leq j < k \leq N$, $1 \leq s < t \leq N$) and containing both \mathcal{F}_{hol} and \mathcal{F}_{ahol} as its subalgebras. The strategy would be then to find a (restriction) morphism $\psi : \tilde{\mathcal{A}}_d \rightarrow \mathcal{F}$ giving \mathcal{F} the structure of a left $\tilde{\mathcal{A}}_d$ –module and afterwards to factorize off the “holomorphic” generators ζ_{jk} and to identify the factor-space $\mathcal{F}/\langle \zeta_{jk} \rangle$ with \mathcal{F}_{ahol} . The morphism ψ is determined by its values on the generators and thus it suffices to prescribe a matrix $\psi(M) \in \text{Mat}(N, \mathcal{F})$ obeying the corresponding relations (2.10). In what follows we shall abuse the notation by writing simply M instead of $\psi(M)$. However, as observed already in Ref. [24], it is not necessary to know the structure of the algebra \mathcal{F} in full detail for successful derivation of the left module structure. A left $\tilde{\mathcal{A}}_d$ –module on \mathcal{F}_{ahol} will be defined by prescribing the action on the unit and then extending it to all polynomials in non-commutative variables ζ_{st}^* with the help of a recursive rule.

Ignoring the fact that we don’t know the commutation relations between the holomorphic and antiholomorphic generators we will introduce projector-like matrices, one for each Grassmann manifold ($1 \leq m \leq N - 1$),

$$\mathfrak{Q}^{(m)} := \begin{pmatrix} \mathbf{I} \\ Z^{(m)*} \end{pmatrix} (\mathbf{I} + Z^{(m)} Z^{(m)*})^{-1} \begin{pmatrix} \mathbf{I} & Z^{(m)} \end{pmatrix}. \quad (4.1)$$

Note that

$$\mathbf{I} - \mathfrak{Q}^{(m)} = \begin{pmatrix} -Z^{(m)} \\ \mathbf{I} \end{pmatrix} (\mathbf{I} + Z^{(m)*} Z^{(m)})^{-1} \begin{pmatrix} -Z^{(m)*} & \mathbf{I} \end{pmatrix}. \quad (4.2)$$

We set also whenever convenient $\mathfrak{Q}^{(0)} := \mathbf{0}$, $\mathfrak{Q}^{(N)} := \mathbf{I}$. It holds true that

$$\mathfrak{Q}^{(m)*} = \mathfrak{Q}^{(m)} \quad \text{and} \quad \mathfrak{Q}^{(m)} \mathfrak{Q}^{(n)} = \mathfrak{Q}^{(n)} \mathfrak{Q}^{(m)} = \mathfrak{Q}^{(\min(m,n))}. \quad (4.3)$$

The second equality means that $(\mathbf{I} - \mathfrak{Q}^{(m)}) \mathfrak{Q}^{(n)} = 0$ for $m \leq n$ and follows immediately from (4.1), (4.2) and (3.24). A morphism $\psi : \tilde{\mathcal{A}}_d \rightarrow \mathcal{F}$ is prescribed by postulating a diagonalization of the quantum positive matrix $M = \Lambda^* \Lambda$,

$$\psi(M) = \xi_1 \mathfrak{Q}^{(1)} + \xi_2 (\mathfrak{Q}^{(2)} - \mathfrak{Q}^{(1)}) + \dots + \xi_N (\mathbf{I} - \mathfrak{Q}^{(N)}).$$

The construction of a left $\tilde{\mathcal{A}}_d$ –module is based on the requirement that this prescription actually defines a homomorphism.

Lemma 4.1. Assume that N real (or real like, i.e., from $\mathbb{R}[[h]] \subset \mathbb{C}[[h]]$) and mutually different parameters ξ_m , $1 \leq m \leq N$, are given. The matrix

$$M = \sum_{m=1}^N \xi_m (\mathfrak{Q}^{(m)} - \mathfrak{Q}^{(m-1)}) \quad (4.4)$$

verifies the relations (repeating (2.10))

$$M_2 R_{12}^{-1} M_1 R_{21}^{-1} = R_{12}^{-1} M_1 R_{21}^{-1} M_2, \quad M^* = M, \quad (4.5)$$

if and only if it holds

$$\mathfrak{Q}_2^{(m)} R_{12}^{-1} M_1 R_{21}^{-1} = R_{12}^{-1} M_1 R_{21}^{-1} \mathfrak{Q}_2^{(m)}, \quad 1 \leq m \leq N-1. \quad (4.6)$$

Proof. Clearly $M^* = M$. Note that (4.3) means

$$(\mathfrak{Q}^{(m)} - \mathfrak{Q}^{(m-1)})(\mathfrak{Q}^{(n)} - \mathfrak{Q}^{(n-1)}) = \delta_{mn}(\mathfrak{Q}^{(m)} - \mathfrak{Q}^{(m-1)}).$$

Multiply the first equality in (4.5) by $(\mathfrak{Q}^{(m)} - \mathfrak{Q}^{(m-1)})$ from the left and by $(\mathfrak{Q}^{(n)} - \mathfrak{Q}^{(n-1)})$ from the right to obtain

$$(\xi_m - \xi_n)(\mathfrak{Q}^{(m)} - \mathfrak{Q}^{(m-1)})R_{12}^{-1}M_1R_{21}^{-1}(\mathfrak{Q}^{(n)} - \mathfrak{Q}^{(n-1)}) = 0.$$

The factor $(\xi_m - \xi_n)$ can be canceled for $m \neq n$. The summation over n , with m being fixed, results in

$$(\mathfrak{Q}^{(m)} - \mathfrak{Q}^{(m-1)})R_{12}^{-1}M_1R_{21}^{-1} = (\mathfrak{Q}^{(m)} - \mathfrak{Q}^{(m-1)})R_{12}^{-1}M_1R_{21}^{-1}(\mathfrak{Q}^{(m)} - \mathfrak{Q}^{(m-1)}).$$

Here the RHS is self-adjoint ($R_{12}^* = R_{21}$) and so the same is true for the LHS. This means that

$$(\mathfrak{Q}^{(n)} - \mathfrak{Q}^{(n-1)})R_{12}^{-1}M_1R_{21}^{-1} = R_{12}^{-1}M_1R_{21}^{-1}(\mathfrak{Q}^{(n)} - \mathfrak{Q}^{(n-1)}). \quad (4.7)$$

To get (4.6) it suffices to sum the equalities (4.7) over n , $1 \leq n \leq m$. The opposite implication is obvious since (4.6) means that $\mathfrak{Q}_2^{(m)}$, $0 \leq m \leq N$, commute with $R_{12}^{-1}M_1R_{21}^{-1}$ and so does M_2 . ■

The commutation relation between $\mathfrak{Q}^{(m)}$ and M implies a commutation relation between $\mathfrak{Z}^{(m)*}$ and M . This will provide us with the desired recursive rule (stated in (4.27) below).

Lemma 4.2. The commutation relation (4.6) is equivalent to

$$(\mathbf{I} - F_2^{(m)} R_{12} \mathfrak{Z}_2^{(m)*} R_{12}^{-1}) M_1 R_{21}^{-1} (E_2^{(m)} + \mathfrak{Z}_2^{(m)*}) = E_2^{(m)} M_1 R_{21}^{-1}. \quad (4.8)$$

Furthermore, the matrix $(\mathbf{I} - F_2^{(m)} R_{12} \mathfrak{Z}_2^{(m)*} R_{12}^{-1})$ is invertible and it holds true that

$$\begin{aligned} (\mathbf{I} - F_2^{(m)} R_{12} \mathfrak{Z}_2^{(m)*} R_{12}^{-1})^{-1} E_2^{(m)} &= R_{12} (E_2^{(m)} + \mathfrak{Z}_2^{(m)*}) R_{21} \\ &\quad - (q - q^{-1}) (E_1^{(m)} + \mathfrak{Z}_1^{(m)*}) R_{12} P(E_1^{(m)} + \mathfrak{Z}_1^{(m)*}). \end{aligned}$$

(4.9)

Proof. In this proof we shall suppress the superscript (m) . (a) The relation (4.6) means that

$$(\mathbf{I} - \mathfrak{Q}_2)R_{12}^{-1}M_1R_{21}^{-1}\mathfrak{Q}_2 = 0.$$

From the structure of \mathfrak{Q} (cf. (4.1), (4.2)) one finds that this is equivalent to

$$(\mathbf{I} - E_2 - \mathfrak{Z}_2^*)R_{12}^{-1}M_1R_{21}^{-1}(E_2 + \mathfrak{Z}_2^*) = 0$$

and hence

$$(F_2 - \mathfrak{Z}_2^*)R_{12}^{-1}M_1R_{21}^{-1}\mathfrak{Z}_2^* = -(F_2 - \mathfrak{Z}_2^*)R_{12}^{-1}M_1R_{21}^{-1}E_2. \quad (4.10)$$

Recall that $\mathfrak{Z}^* = F\mathfrak{Z}^*E$. Thus the both sides of (4.10) are invariant with respect to multiplication by F_2 from the left. Since (cf. (2.15))

$$F_2R_{12}^{\pm 1}F_2R_{12}^{\mp 1} = F_2,$$

the multiplication by F_2R_{12} from the left results in an equivalent equality, namely

$$(F_2 - F_2R_{12}\mathfrak{Z}_2^*R_{12}^{-1})M_1R_{21}^{-1}\mathfrak{Z}_2^* = -(F_2 - F_2R_{12}\mathfrak{Z}_2^*R_{12}^{-1})M_1R_{21}^{-1}E_2. \quad (4.11)$$

Now it suffices to observe that $F_2R_{21}^{-1}\mathfrak{Z}_2^* = R_{21}^{-1}\mathfrak{Z}_2^*$ and $F_2R_{21}^{-1}E_2 = R_{21}^{-1}E_2 - E_2R_{21}^{-1}$ in order to conclude that (4.11) coincides with (4.8).

(b) In the lexicographically ordered basis, $R_{12}^{\pm 1}$ is lower triangular ($R_{jk,st} = 0$ for $j < s$ and $R_{jk,jt} = \delta_{kt}R_{jk,jt}$). It follows readily that $F_2R_{12}\mathfrak{Z}_2^*R_{12}^{-1}$ is lower triangular with vanishing diagonal and hence nilpotent and consequently $(\mathbf{I} - F_2R_{12}\mathfrak{Z}_2^*R_{12}^{-1})$ is invertible. It remains to show that

$$E_2 = (\mathbf{I} - F_2R_{12}\mathfrak{Z}_2^*R_{12}^{-1})[R_{12}(E_2 + \mathfrak{Z}_2^*)R_{21} - (q - q^{-1})(E_1 + \mathfrak{Z}_1^*)R_{12}P(E_1 + \mathfrak{Z}_1^*)].$$

We have for the LHS (cf. (2.15), (2.4), (2.17))

$$\begin{aligned} E_2 &= E_2R_{21}^{-1}(E_2 + \mathfrak{Z}_2^*)R_{21} \\ &= E_2R_{12}(E_2 + \mathfrak{Z}_2^*)R_{21} - (q - q^{-1})E_2(E_1 + \mathfrak{Z}_1^*)R_{12}P(E_1 + \mathfrak{Z}_1^*) \end{aligned}$$

while the RHS can be expanded and treated using (3.23) and

$$F_2R_{12}\mathfrak{Z}_2^* = F_2R_{12}(E_2 + \mathfrak{Z}_2^*), \quad (E + \mathfrak{Z}^*)^2 = (E + \mathfrak{Z}^*).$$

This gives immediately the result. ■

In addition to the recursive rule we also need to know the action on the unit, i.e., the expression $M \cdot 1$. Thinking of \mathcal{F}_{ahol} as the factor space $\mathcal{F}/\langle \zeta_{jk} \rangle$ we require naturally that $Z^{(m)} \cdot 1 = 0$, $\forall m$. This implies

$$\mathfrak{Q}^{(m)} \cdot 1 = \begin{pmatrix} \mathbf{I} \\ Z^{(m)*} \end{pmatrix} (\mathbf{I} + Z^{(m)}Z^{(m)*})^{-1} \begin{pmatrix} \mathbf{I} & \mathbf{0} \end{pmatrix} \cdot 1.$$

Guided by the experience obtained when working with quantum Grassmannians [6] we accept as an ansatz that for some scalar η_m ,

$$(\mathbf{I} + Z^{(m)} Z^{(m)*})^{-1} \cdot 1 = \eta_m \mathbf{I},$$

whence

$$\mathfrak{Q}^{(m)} \cdot 1 = \eta_m (E^{(m)} + \mathfrak{Z}^{(m)*}). \quad (4.12)$$

To get rid of superfluous parameters we use the substitution

$$\lambda_N := \xi_N, \quad \lambda_{N-j} := \lambda_{N-j+1} + (\xi_{N-j} - \xi_{N-j+1})\eta_{N-j} \quad \text{for } j = 1, \dots, N-1. \quad (4.13)$$

Now we are ready to state the result provided \mathcal{F}_{ahol} is described in terms of $\mathcal{G}_{ahol}^{(m)}$, $1 \leq m \leq N-1$, as given in Proposition 3.3. Before let us formulate some auxiliary relations needed for the proof. At the same time, we introduce some shorthand notation, also only for the purpose of this proof. Set

$$\gamma := q - q^{-1}, \quad (4.14)$$

$$\mathfrak{X}^{(m)} := E^{(m)} + \mathfrak{Z}^{(m)*}, \quad (4.15)$$

$$X_{12}^{(m)} := (\mathbf{I} - F_2^{(m)} R_{12} \mathfrak{Z}_2^{(m)*} R_{12}^{-1})^{-1} E_2^{(m)}, \quad (4.16)$$

and $X_{12}^{(0)} := \mathbf{0}$, $X_{12}^{(N)} := \mathbf{I}$. Thus, by Proposition 3.3 and (3.25), we have

$$m \leq n \implies \mathfrak{X}^{(n)} \mathfrak{X}^{(m)} = \mathfrak{X}^{(m)} \mathfrak{X}^{(n)} = \mathfrak{X}^{(m)}, \quad (4.17)$$

$$m \leq n \implies R_{21} \mathfrak{X}_1^{(m)} R_{12} \mathfrak{X}_2^{(n)} = \mathfrak{X}_2^{(n)} R_{21} \mathfrak{X}_1^{(m)} R_{12}, \quad (4.18)$$

and by Lemma 4.2,

$$X_{12}^{(m)} = R_{12} \mathfrak{X}_2^{(m)} R_{21} - \gamma \mathfrak{X}_1^{(m)} R_{12} P \mathfrak{X}_1^{(m)}. \quad (4.19)$$

Lemma 4.3. *For $1 \leq m \leq n \leq N$, it holds true that*

$$X_{12}^{(n)} X_{12}^{(m)} = X_{12}^{(m)}, \quad (4.20)$$

$$R_{32} X_{12}^{(m)} R_{23} X_{13}^{(n)} = X_{13}^{(n)} R_{32} X_{12}^{(m)} R_{23}, \quad (4.21)$$

$$(X_{21}^{(m)} \mathfrak{X}_2^{(n)} + X_{21}^{(n)} \mathfrak{X}_2^{(m)}) R_{12}^{-1} = R_{12}^{-1} (X_{12}^{(m)} \mathfrak{X}_1^{(n)} + X_{12}^{(n)} \mathfrak{X}_1^{(m)}). \quad (4.22)$$

Proof. (4.20): Observe that

$$X_{12}^{(m)} = R_{12} \mathfrak{X}_2^{(m)} R_{21} (\mathbf{I} - \gamma \mathfrak{X}_1^{(m)} R_{21}^{-1} P_{12}) \quad (4.23)$$

and ($m \leq n$)

$$(\mathbf{I} - \gamma \mathfrak{X}_1^{(n)} R_{21}^{-1} P_{12}) R_{12} \mathfrak{X}_2^{(m)} R_{21} = R_{21}^{-1} \mathfrak{X}_2^{(m)} R_{21}.$$

Consequently,

$$X_{12}^{(n)} X_{12}^{(m)} = R_{12} \mathfrak{X}_2^{(n)} R_{21} \cdot R_{21}^{-1} \mathfrak{X}_2^{(m)} R_{21} (\mathbf{I} - \gamma \mathfrak{X}_1^{(m)} R_{21}^{-1} P_{12}) = X_{12}^{(m)}.$$

(4.21): Using repeatedly YB equation, (4.18) and (4.17), one can derive that

$$\begin{aligned}\mathfrak{X}_3^{(n)} R_{13}^{-1} R_{32} (R_{12} \mathfrak{X}_2^{(m)} R_{21}) R_{23} R_{13} \mathfrak{X}_3^{(n)} &= \mathfrak{X}_3^{(n)} R_{12} R_{32} \mathfrak{X}_2^{(m)} R_{23} R_{21} \mathfrak{X}_3^{(n)} \\ &= R_{13}^{-1} R_{32} (R_{12} \mathfrak{X}_2^{(m)} R_{21}) R_{23} R_{13} \mathfrak{X}_3^{(n)},\end{aligned}$$

and (making use also of $R_{13}^{-1} = R_{31} - \gamma P_{13}$)

$$\begin{aligned}\mathfrak{X}_3^{(n)} R_{13}^{-1} R_{32} (\mathfrak{X}_1^{(m)} R_{12} P_{12} \mathfrak{X}_1^{(m)}) R_{23} R_{13} \mathfrak{X}_3^{(n)} &= \mathfrak{X}_3^{(n)} R_{13}^{-1} \mathfrak{X}_1^{(m)} R_{13} P_{12} R_{21} R_{31} \mathfrak{X}_1^{(m)} R_{13} \mathfrak{X}_3^{(n)} \\ &= R_{13}^{-1} R_{32} (\mathfrak{X}_1^{(m)} R_{12} P_{12} \mathfrak{X}_1^{(m)}) R_{23} R_{13} \mathfrak{X}_3^{(n)}.\end{aligned}$$

In virtue of (4.19), this means jointly that

$$\mathfrak{X}_3^{(n)} R_{13}^{-1} R_{32} X_{12}^{(m)} R_{23} R_{13} \mathfrak{X}_3^{(n)} = R_{13}^{-1} R_{32} X_{12}^{(m)} R_{23} R_{13} \mathfrak{X}_3^{(n)},$$

and consequently, utilizing once more (4.23),

$$(F_3^{(n)} - F_3^{(n)} R_{13} \mathfrak{X}_3^{(n)} R_{13}^{-1}) R_{32} X_{12}^{(m)} R_{23} X_{13}^{(n)} = 0. \quad (4.24)$$

Furthermore,

$$\begin{aligned}E_3^{(n)} R_{32} X_{12}^{(m)} R_{23} X_{13}^{(n)} &= E_3^{(n)} R_{32} X_{12}^{(m)} E_2^{(n)} E_3^{(n)} R_{23} X_{13}^{(n)} \\ &= E_3^{(n)} R_{32} X_{12}^{(m)} R_{23},\end{aligned} \quad (4.25)$$

since $E_3^{(n)} X_{13}^{(n)} = E_3^{(n)}$. Summing (4.24) and (4.25) we get the sought equality in the form $(F_3^{(n)} R_{13} \mathfrak{X}_3^{(n)} = F_3^{(n)} R_{13} \mathfrak{Z}_3^{(n)*})$

$$(\mathbf{I} - F_3^{(n)} R_{13} \mathfrak{Z}_3^{(n)*} R_{13}^{-1}) R_{32} X_{12}^{(m)} R_{23} X_{13}^{(n)} = E_3^{(n)} R_{32} X_{12}^{(m)} R_{23}.$$

(4.22): Using again (4.19), (4.17) and (4.18) as well as (2.4) we find that

$$\begin{aligned}\text{LHS} &= R_{21} \mathfrak{X}_1^{(m)} R_{12} \mathfrak{X}_2^{(n)} R_{12}^{-1} - \gamma \mathfrak{X}_2^{(m)} R_{21} P \mathfrak{X}_2^{(m)} R_{12}^{-1} \\ &\quad + R_{21} \mathfrak{X}_1^{(n)} R_{12} \mathfrak{X}_2^{(m)} R_{12}^{-1} - \gamma \mathfrak{X}_2^{(n)} R_{21} P \mathfrak{X}_2^{(m)} R_{12}^{-1} \\ &= \mathfrak{X}_2^{(n)} R_{21} \mathfrak{X}_1^{(m)} - \gamma R_{12}^{-1} \mathfrak{X}_1^{(m)} R_{12} \mathfrak{X}_2^{(m)} P + \mathfrak{X}_2^{(m)} R_{21} \mathfrak{X}_1^{(n)} R_{21}^{-1} R_{12}^{-1},\end{aligned}$$

and

$$\text{RHS} = \mathfrak{X}_2^{(m)} R_{21} \mathfrak{X}_1^{(n)} - \gamma R_{12}^{-1} \mathfrak{X}_1^{(m)} R_{12} P \mathfrak{X}_1^{(m)} + \mathfrak{X}_2^{(n)} R_{21} \mathfrak{X}_1^{(m)} - \gamma R_{12}^{-1} \mathfrak{X}_1^{(n)} R_{12} P \mathfrak{X}_1^{(m)}.$$

Consequently,

$$\text{LHS} - \text{RHS} = \mathfrak{X}_2^{(m)} R_{21} \mathfrak{X}_1^{(n)} R_{21}^{-1} (R_{12}^{-1} - R_{21}) + \gamma R_{12}^{-1} \mathfrak{X}_1^{(n)} R_{12} \mathfrak{X}_2^{(m)} P = 0. \quad \blacksquare$$

Proposition 4.4. *The relations*

$$M \cdot 1 = (\lambda_1 - \lambda_2)(E^{(1)} + \mathfrak{Z}^{(1)*}) + \dots + (\lambda_{N-1} - \lambda_N)(E^{(N-1)} + \mathfrak{Z}^{(N-1)*}) + \lambda_N \mathbf{I},$$

$$(4.26)$$

$$M_1 R_{21}^{-1} \cdot (E_2^{(m)} + \mathfrak{Z}_2^{(m)*}) f = (\mathbf{I} - F_2^{(m)} R_{12} \mathfrak{Z}_2^{(m)*} R_{12}^{-1})^{-1} E_2^{(m)} M_1 R_{21}^{-1} \cdot f, \quad (4.27)$$

for all $f \in \mathcal{F}_{ahol}$, $1 \leq m \leq N-1$, define on \mathcal{F}_{ahol} unambiguously the structure of a left $\tilde{\mathcal{A}}_d$ -module (the central dot “ \cdot ” stands for the action) depending on N scalar parameters $\lambda_1, \dots, \lambda_N$.

Proof. As already mentioned, the idea of defining a module this way was utilized in Ref. [24] (Proposition 5.4) and the proof is quite similar, too. Here we rely heavily on Proposition 3.3 giving a description of \mathcal{F}_{ahol} in terms of $\mathcal{G}_{ahol}^{(m)}$. First we have to show that (4.26) and (4.27) define correctly a linear mapping $\mathcal{F}_{ahol} \rightarrow \text{Mat}(N, \mathcal{F}_{ahol}) : f \mapsto M \cdot f$. Let $\bar{\mathcal{F}}$ be the free algebra generated by $\bar{z}_{jk}^{(m)}$, $1 \leq j \leq m < k \leq N$, and the matrix $\bar{\mathfrak{Z}}^{(m)}$ be obtained from $\mathfrak{Z}^{(m)*}$ when replacing the entries $z_{jk}^{(m)*}$ by $\bar{z}_{jk}^{(m)}$. Furthermore, we use in an obvious sense the symbols $\bar{\mathfrak{X}}^{(m)}$ and $\bar{X}_{12}^{(m)}$ parallelly to (4.15) and (4.16), respectively. Hence \mathcal{F}_{ahol} is obtained from $\bar{\mathcal{F}}$ by means of factorization by the two-sided ideal generated by the relations (4.17), (4.18), with $\mathfrak{X}^{(m)}$'s being replaced by $\bar{\mathfrak{X}}^{(m)}$'s, and the elements $z_{jk}^{(m)*}$ are the factor images of $\bar{z}_{jk}^{(m)}$. Doing the same replacement in (4.26) and (4.27) one obtains a well defined linear mapping

$$\bar{\mathcal{F}} \rightarrow \text{Mat}(N, \bar{\mathcal{F}}) : f \mapsto M \cdot f. \quad (4.28)$$

A straightforward calculation gives for $m \leq n$ and $\forall f \in \bar{\mathcal{F}}$,

$$\begin{aligned} M_1 R_{21}^{-1} \cdot (\bar{\mathfrak{X}}_2^{(n)} \bar{\mathfrak{X}}_2^{(m)} - \bar{\mathfrak{X}}_2^{(m)}) f &= (\bar{X}_{12}^{(n)} \bar{X}_{12}^{(m)} - \bar{X}_{12}^{(m)}) M_1 R_{21}^{-1} \cdot f, \\ M_1 R_{31}^{-1} R_{21}^{-1} \cdot (R_{32} \bar{\mathfrak{X}}_2^{(m)} R_{23} \bar{\mathfrak{X}}_3^{(n)} - \bar{\mathfrak{X}}_3^{(n)} R_{32} \bar{\mathfrak{X}}_2^{(m)} R_{23}) f \\ &= (R_{32} \bar{X}_{12}^{(m)} R_{23} \bar{X}_{13}^{(n)} - \bar{X}_{13}^{(n)} R_{32} \bar{X}_{12}^{(m)} R_{23}) M_1 R_{31}^{-1} R_{21}^{-1} \cdot f. \end{aligned}$$

This means that the linear mapping (4.28) factorizes from $\bar{\mathcal{F}}$ to \mathcal{F}_{ahol} if and only if the factor-images of the matrices $(\bar{X}_{12}^{(n)} \bar{X}_{12}^{(m)} - \bar{X}_{12}^{(m)})$ and $(R_{32} \bar{X}_{12}^{(m)} R_{23} \bar{X}_{13}^{(n)} - \bar{X}_{13}^{(n)} R_{32} \bar{X}_{12}^{(m)} R_{23})$ vanish. But these are exactly the relations (4.20) and (4.21) proven in Lemma 4.3.

To show that \mathcal{F}_{ahol} is really a left $\tilde{\mathcal{A}}_d$ -module we have to verify the equality

$$M_2 R_{12}^{-1} M_1 R_{21}^{-1} \cdot 1 = R_{12}^{-1} M_1 R_{21}^{-1} M_2 \cdot 1 \quad (4.29)$$

and the implication

$$\begin{aligned} (M_2 R_{12}^{-1} M_1 R_{21}^{-1}) R_{31}^{-1} R_{32}^{-1} \cdot f &= (R_{12}^{-1} M_1 R_{21}^{-1} M_2) R_{31}^{-1} R_{32}^{-1} \cdot f \implies \\ (M_2 R_{12}^{-1} M_1 R_{21}^{-1}) R_{31}^{-1} R_{32}^{-1} \cdot \mathfrak{Z}_3^{(m)*} f &= (R_{12}^{-1} M_1 R_{21}^{-1} M_2) R_{31}^{-1} R_{32}^{-1} \cdot \mathfrak{Z}_3^{(m)*} f, \end{aligned} \quad (4.30)$$

for $\forall f \in \mathcal{F}_{ahol}$, $1 \leq m \leq N-1$, since then (4.29) and (4.30) jointly imply

$$M_2 R_{12}^{-1} M_1 R_{21}^{-1} \cdot f = R_{12}^{-1} M_1 R_{21}^{-1} M_2 \cdot f, \quad \forall f \in \mathcal{F}_{ahol}.$$

Verification of (4.29). Using (4.26) and (4.27) one finds easily that (4.29) means ($\lambda_{N+1} := 0$)

$$\begin{aligned} & \sum_{m=1}^N (\lambda_m - \lambda_{m+1}) X_{21}^{(m)} \sum_{n=1}^N (\lambda_n - \lambda_{n+1}) \mathfrak{X}_2^{(n)} R_{12}^{-1} R_{21}^{-1} \\ &= R_{12}^{-1} \sum_{m=1}^N (\lambda_m - \lambda_{m+1}) X_{12}^{(m)} \sum_{n=1}^N (\lambda_n - \lambda_{n+1}) \mathfrak{X}_1^{(n)} R_{21}^{-1}. \end{aligned}$$

But this equality follows immediately from the relation (4.22) proven in Lemma 4.3.

Verification of (4.30). Note first that the equality after the sign of implication in (4.30) can be replaced by

$$(M_2 R_{12}^{-1} M_1 R_{21}^{-1}) R_{31}^{-1} R_{32}^{-1} \cdot \mathfrak{X}_3^{(m)} f = (R_{12}^{-1} M_1 R_{21}^{-1} M_2) R_{31}^{-1} R_{32}^{-1} \cdot \mathfrak{X}_3^{(m)} f,$$

Furthermore, reversing the proof of Lemma 4.2, part (a), one finds that (4.27) is equivalent to

$$R_{12}^{-1} M_1 R_{21}^{-1} \cdot \mathfrak{X}_2^{(m)} f = \mathfrak{X}_2^{(m)} R_{12}^{-1} M_1 R_{21}^{-1} \cdot \mathfrak{X}_2^{(m)} f, \quad \forall f \in \mathcal{F}_{ahol}. \quad (4.31)$$

Using repeatedly YB equation and (4.31) one derives that

$$\begin{aligned} & (\mathbf{I} - \mathfrak{X}_3^{(m)}) R_{23}^{-1} R_{13}^{-1} (M_2 R_{12}^{-1} M_1 R_{21}^{-1}) R_{31}^{-1} R_{32}^{-1} \cdot \mathfrak{X}_3^{(m)} f \\ &= (\mathbf{I} - \mathfrak{X}_3^{(m)}) (R_{23}^{-1} M_2 R_{32}^{-1}) R_{12}^{-1} (R_{13}^{-1} M_1 R_{31}^{-1}) \cdot \mathfrak{X}_3^{(m)} R_{21}^{-1} f \\ &= 0, \end{aligned} \quad (4.32)$$

for $(\mathbf{I} - \mathfrak{X}_3^{(m)}) \mathfrak{X}_3^{(m)} = 0$. It follows from (4.32) that

$$\begin{aligned} & (\mathbf{I} - F_3^{(m)} R_{13} R_{23} \mathfrak{X}_3^{(m)} R_{23}^{-1} R_{13}^{-1}) (M_2 R_{12}^{-1} M_1 R_{21}^{-1}) R_{31}^{-1} R_{32}^{-1} \cdot \mathfrak{X}_3^{(m)} f \\ &= E_3^{(m)} (M_2 R_{12}^{-1} M_1 R_{21}^{-1}) R_{31}^{-1} R_{32}^{-1} \cdot f, \end{aligned} \quad (4.33)$$

for $E_3^{(m)} R_{31}^{-1} R_{32}^{-1} \mathfrak{X}_3^{(m)} = E_3^{(m)} R_{31}^{-1} R_{32}^{-1}$. Quite similarly one obtains

$$\begin{aligned} & (\mathbf{I} - F_3^{(m)} R_{13} R_{23} \mathfrak{X}_3^{(m)} R_{23}^{-1} R_{13}^{-1}) (R_{12}^{-1} M_1 R_{21}^{-1} M_2) R_{31}^{-1} R_{32}^{-1} \cdot \mathfrak{X}_3^{(m)} f \\ &= E_3^{(m)} (R_{12}^{-1} M_1 R_{21}^{-1} M_2) R_{31}^{-1} R_{32}^{-1} \cdot f, \end{aligned} \quad (4.34)$$

The right hand sides of (4.33) and (4.34) are equal by assumption and so, to complete the proof, it suffices to show that the matrix $(\mathbf{I} - F_3^{(m)} R_{13} R_{23} \mathfrak{X}_3^{(m)} R_{23}^{-1} R_{13}^{-1})$ is invertible. But using a similar argument as in the proof of Lemma 4.2, part (b), one finds that the matrix

$$F_3^{(m)} R_{13} R_{23} \mathfrak{X}_3^{(m)} R_{23}^{-1} R_{13}^{-1} = F_3^{(m)} R_{13} R_{23} \mathfrak{Z}_3^{(m)*} R_{23}^{-1} R_{13}^{-1}$$

is nilpotent. ■

5 LEIBNIZ RULE AND THE DRESSING TRANSFORMATION

The right dressing transformation

$$\mathcal{R} : \mathcal{A}_q(AN) \rightarrow \mathcal{A}_q(AN) \otimes \mathcal{A}_q(SU(N))$$

can be introduced formally using the canonical element in $\mathcal{A}_q(AN) \otimes \mathcal{A}_q(SU(N))$ [29, 7] and it factorizes from $\mathcal{A}_q(AN)$ to both \mathcal{F}_{hol} and \mathcal{F}_{ahol} . Dually it induces a left action of $\mathcal{U}_h(\mathfrak{su}(N))$ on $\mathcal{A}_q(AN)$ (or \mathcal{F}_{hol} or \mathcal{F}_{ahol}) via the pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{U}_h(\mathfrak{su}(N))$ and $\mathcal{A}_q(SU(N))$,

$$\xi_Y \cdot f := (\text{id} \otimes \langle Y, \cdot \rangle) \mathcal{R}(f). \quad (5.1)$$

Leibniz rule for ξ means that

$$\xi_Y \cdot fg = (\xi_{Y_{(1)}} \cdot f)(\xi_{Y_{(2)}} \cdot g), \quad \text{with } \Delta Y = Y_{(1)} \otimes Y_{(2)}. \quad (5.2)$$

The aim of this section is to show that Leibniz rule for ξ (acting on \mathcal{F}_{ahol}) induces the recursive rule (4.27).

Here we introduce the dressing transformation

$$\mathcal{R} : \mathcal{F}_{hol} \rightarrow \mathcal{F}_{hol} \otimes \mathcal{A}_q(SU(N)) \quad (5.3)$$

directly by prescribing its values on the generators ζ_{jk} arranged in the matrix \mathcal{Z} . As usual, the identifications $\mathcal{F}_{hol} \equiv \mathcal{F}_{hol} \otimes 1$ and $\mathcal{A}_q(SU(N)) \equiv 1 \otimes \mathcal{A}_q(SU(N))$ simplify a lot the notation. We define, with the help of Gauss decomposition and formally in the same way as classically,

$$\mathcal{R}(\mathcal{Z}) := (\mathcal{Z}U)_+, \quad \text{where } \mathcal{Z}U = (\mathcal{Z}U)_- (\mathcal{Z}U)_+ \quad (5.4)$$

and $(\mathcal{Z}U)_+$ is upper triangular with units on the diagonal while $(\mathcal{Z}U)_-$ is lower triangular (U still designates the vector corepresentation of $\mathcal{A}_q(SU(N))$). \mathcal{R} extends to an algebra homomorphism and it holds

$$(\text{id} \otimes \varepsilon) \circ \mathcal{R} = \text{id}, \quad (5.5)$$

$$(\mathcal{R} \otimes \text{id}) \circ \mathcal{R} = (\text{id} \otimes \Delta) \circ \mathcal{R}, \quad (5.6)$$

as follows readily from (2.1) and the uniqueness of Gauss decomposition.

Let us rewrite the dressing transformation (5.4) in terms of coordinate functions $z_{jk}^{(m)}$ on quantum Grassmannians. From (3.20) one finds that

$$\begin{aligned} E^{(m)} \mathcal{Z} U \mathcal{R}(E^{(m)} + \mathfrak{Z}^{(m)}) &= E^{(m)} \mathcal{Z} U \mathcal{R}(\mathcal{Z})^{-1} E^{(m)} \mathcal{R}(\mathcal{Z}) \\ &= E^{(m)} (\mathcal{Z}U)_- (\mathbf{I} - F^{(m)}) (\mathcal{Z}U)_+ \\ &= E^{(m)} \mathcal{Z}U, \end{aligned}$$

for $E^{(m)}(\mathcal{Z}U)_-F^{(m)} = 0$ owing to the fact that $(\mathcal{Z}U)_-$ is lower triangular. Similarly, $F^{(m)}\mathcal{R}(\mathcal{Z})^{-1}E^{(m)} = 0$ and thus

$$(\mathcal{Z}F^{(m)} + E^{(m)}\mathcal{Z}U)\mathcal{R}(E^{(m)} + \mathfrak{Z}^{(m)}) = E^{(m)}\mathcal{Z}U.$$

Consequently,

$$\begin{aligned}\mathcal{R}(E^{(m)} + \mathfrak{Z}^{(m)}) &= (F^{(m)} + \mathcal{Z}^{-1}E^{(m)}\mathcal{Z}U)^{-1}\mathcal{Z}^{-1}E^{(m)}\mathcal{Z}U \\ &= [F^{(m)} + (E^{(m)} + \mathfrak{Z}^{(m)})U]^{-1}(E^{(m)} + \mathfrak{Z}^{(m)})U.\end{aligned}\quad (5.7)$$

This can be expressed also in terms of blocks $Z^{(m)}$. Decompose U into blocks,

$$U = \begin{pmatrix} A^{(m)} & B^{(m)} \\ C^{(m)} & D^{(m)} \end{pmatrix},$$

where $A^{(m)}$ has dimension $m \times m$, $B^{(m)}$ has dimension $m \times (N - m)$ etc. The result is

$$\mathcal{R}(Z^{(m)}) = (A^{(m)} + Z^{(m)}C^{(m)})^{-1}(B^{(m)} + Z^{(m)}D^{(m)})$$

and this is the correct formula [6]. The dressing transformation on \mathcal{F}_{ahol} is obtained readily by taking the adjoints. Particularly,

$$\mathcal{R}(E^{(m)} + \mathfrak{Z}^{(m)*}) = U^*(E^{(m)} + \mathfrak{Z}^{(m)*})[F^{(m)} + U^*(E^{(m)} + \mathfrak{Z}^{(m)*})]^{-1}. \quad (5.8)$$

Let us add a couple of remarks. Despite of the fact that the formulas (5.4), (5.7) contain rational singularities (a consequence of the localization from the dressing orbit to the big cell) the action ξ (local in its nature) is free of any singularities. Furthermore, observe from (4.1) and

$$\mathcal{R}(\mathbf{I} \quad Z^{(m)}) = (A^{(m)} + Z^{(m)}C^{(m)})^{-1}(\mathbf{I} \quad Z^{(m)})U$$

that $\mathcal{R}(\mathfrak{Q}^{(m)}) = U^*\mathfrak{Q}^{(m)}U$ and thus the quantum diagonalization (4.4) is in accordance with the rule $\mathcal{R}(\Lambda^*\Lambda) = U^*\Lambda^*\Lambda U$, as it should be [29, 6]. Finally, recall that the comultiplication Δ is always assumed to come from $\mathcal{U}_h(\mathfrak{su}(N))$ rather than from $\mathcal{A}_q(AN)$. Now we can formulate a statement relating Leibniz rule to the $\tilde{\mathcal{A}}_d$ -module \mathcal{F}_{ahol} which was defined in Proposition 4.4.

Proposition 5.1. *For $\forall Y \in \tilde{\mathcal{A}}_d$ and $\forall \psi, f \in \mathcal{F}_{ahol}$, it holds*

$$Y \cdot \psi f = (\xi_{Y_{(1)}} \cdot \psi)(Y_{(2)} \cdot f), \quad \text{with } \Delta Y = Y_{(1)} \otimes Y_{(2)}. \quad (5.9)$$

Remark. This is in accordance with the classical case. The method of orbits, if expressed in local coordinates, yields a representation of Lie algebra $\mathfrak{su}(N)$ in terms of first order differential operators,

$$Y \in \mathfrak{su}(N) \mapsto p_Y(\bar{z})\partial_{\bar{z}} + q_Y(\bar{z}),$$

with generally non-vanishing zero-order term $q_Y(\bar{z})$ and with the first-order term identical to the vector field $\xi_Y = p_Y(\bar{z})\partial_{\bar{z}}$ coming from the infinitesimal coadjoint action. Clearly,

$$Y \cdot 1 = q_Y(\bar{z}) \quad \text{and} \quad Y \cdot \psi f = (\xi_Y \cdot \psi)f + \psi(Y \cdot f).$$

Proof. As we are facing two actions ($\xi_X \xi_Y = \xi_{XY}$ and $X \cdot (Y \cdot f) = (XY) \cdot f$) it is sufficient to verify (5.9) only for the generators of $\tilde{\mathcal{A}}_d$, i.e., for Y running over the entries of M . Note that (5.9) holds trivially for $Y = 1$ ($\Delta 1 = 1 \otimes 1$) and the same is true for $\psi = 1$ owing to the equalities $\varepsilon(Y_{(1)})Y_{(2)} = Y$ and

$$\xi_Y \cdot 1 = (\text{id} \otimes \langle Y, \cdot \rangle)1 \otimes 1 = \varepsilon(Y).$$

In virtue of Leibniz rule (5.2) and the co-associativity of Δ it suffices, too, to verify (5.9) for ψ running over the generators of \mathcal{F}_{ahol} .

Observe that

$$M_2 = \Lambda_2^* \Lambda_2 = \text{tr}_1(P_{12} \Lambda_1^* \Lambda_2),$$

where tr_1 means the trace applied only in the first factor of the tensor product in question. Thus we can formulate the problem in the following way. Evaluate

$$\Lambda_1^* \Lambda_2 R_{32}^{-1} \cdot (E_3^{(m)} + \mathfrak{Z}_3^{(m)*})f$$

using the rule (5.9) and then apply to the obtained expression

$$(\mathbf{I} - F_3^{(m)} R_{23} \mathfrak{Z}_3^{(n)*} R_{23}^{-1}) \text{tr}_1(P_{12} \cdot).$$

The result should coincide with that one given by the recursive rule (4.27), i.e., with $E_3^{(m)} M_2 R_{32}^{-1} \cdot f$. In the rest of the proof we drop the superscript (m) .

As the comultiplication Δ_{AN} in (2.8) is opposite to Δ we have

$$\Delta \Lambda^t = \Lambda^t \dot{\otimes} \Lambda^t, \quad \Delta \Lambda^* = \Lambda^* \dot{\otimes} \Lambda^*.$$

Thus we start from (cf. (5.8))

$$\begin{aligned} & (\langle \Lambda_1^* \Lambda_2^t, \cdot \rangle \times \Lambda_1^* \Lambda_2^t)^{t_2} \left(R_{32}^{-1} U_3^* (E_3 + \mathfrak{Z}_3^*) [F_3 + U_3^* (E_3 + \mathfrak{Z}_3^*)]^{-1} \times f \right) \\ &= \{ (R_{32}^{-1})^{t_2} \langle \Lambda_1^* \Lambda_2^t, \cdot \rangle U_3^* (E_3 + \mathfrak{Z}_3^*) [F_3 + U_3^* (E_3 + \mathfrak{Z}_3^*)]^{-1} \Lambda_1^* \Lambda_2^t \cdot f \}^{t_2}. \end{aligned} \quad (5.10)$$

The pairing $\langle \Lambda_1^* \Lambda_2^t, \cdot \rangle$ acts on the elements of algebra $\mathcal{A}_q(SU(N))$ occuring in the matrices U_3^* . In view of (2.14), we have

$$\langle \Lambda_1^* \Lambda_2^t, U_3^* \rangle = \langle \Lambda_1^* \Lambda_2^t, U_3 \rangle^{-1} = ((R_{32}^{-1})^{t_2})^{-1} R_{13}.$$

Consequently, the expression (5.10) equals

$$\{ R_{13} (E_3 + \mathfrak{Z}_3^*) [F_3 + ((R_{32}^{-1})^{t_2})^{-1} R_{13} (E_3 + \mathfrak{Z}_3^*)]^{-1} \Lambda_1^* \Lambda_2^t \cdot f \}^{t_2}.$$

When applying $\text{tr}_1(P_{12}\cdot)$ observe that (for any X_{123})

$$\text{tr}_1(P_{12}(X_{123}\Lambda_1^*\Lambda_2^t)^{t_2}) = \text{tr}_1(P_{12}X_{123}^{t_2}M_1)$$

and when multiplying by $(\mathbf{I} - F_3R_{23}\mathfrak{Z}_3^*R_{23}^{-1})$ from the left note that

$$\begin{aligned} (\mathbf{I} - F_3R_{23}\mathfrak{Z}_3^*R_{23}^{-1})R_{23}(E_3 + \mathfrak{Z}_3^*) &= R_{23}(E_3 + \mathfrak{Z}_3^*) - F_3R_{23}(E_3 + \mathfrak{Z}_3^*) \\ &= E_3R_{23}(E_3 + \mathfrak{Z}_3^*). \end{aligned}$$

This way we arrive at the expression

$$\text{tr}_1(P_{12}\{E_3R_{13}(E_3 + \mathfrak{Z}_3^*)[F_3 + ((R_{32}^{-1})^{t_2})^{-1}R_{13}(E_3 + \mathfrak{Z}_3^*)]^{-1}\}^{t_2}M_1 \cdot f). \quad (5.11)$$

Finally we use (cf. (2.15))

$$E_3 = E_3(R_{32}^{-1})^{t_2}E_3((R_{32}^{-1})^{t_2})^{-1}$$

and the obvious equality (multiply by $[\dots]$ from the right)

$$E_3((R_{32}^{-1})^{t_2})^{-1}R_{13}(E_3 + \mathfrak{Z}_3^*)[F_3 + ((R_{32}^{-1})^{t_2})^{-1}R_{13}(E_3 + \mathfrak{Z}_3^*)]^{-1} = E_3$$

to conclude that (5.11) equals

$$\begin{aligned} \text{tr}_1(P_{12}\{E_3(R_{32}^{-1})^{t_2}E_3\}^{t_2}M_1 \cdot f) &= E_3 \text{tr}_1(P_{12}R_{32}^{-1}M_1 \cdot f) \\ &= E_3M_2R_{32}^{-1} \cdot f, \end{aligned}$$

as required. ■

Remark. Let τ_λ be an irreducible finite-dimensional representation of $\mathcal{U}_h(\mathfrak{su}(N))$ acting in a Hilbert space \mathcal{H}_λ and corresponding to a lowest weight λ with a lowest weight vector e_λ . Let $\mathcal{T}^\lambda \in \text{End}(\mathcal{H}_\lambda) \otimes \mathcal{A}_q(SU(N))$ be the related corepresentation of $\mathcal{A}_q(SU(N))$. As shown in Ref. [7], basically all information about the representation is encoded in the element

$$w_\lambda := ((e_\lambda, \cdot e_\lambda) \otimes \text{id})\mathcal{T}^\lambda \in \mathcal{A}_q(SU(N)).$$

One can naturally realize the algebras \mathcal{F}_{hol} and \mathcal{F}_{ahol} (but not the full algebra \mathcal{F} of real analytic functions) as subalgebras in the localization of $\mathcal{A}_q(SU(N))$ when the element w_λ is allowed to be invertible. The comultiplication in $\mathcal{A}_q(SU(N))$, being restricted to \mathcal{F}_{ahol} , coincides with the dressing transformation. This means also that $\Delta(\mathcal{F}_{ahol}) \subset \mathcal{F}_{ahol} \otimes \mathcal{A}_q(SU(N))$. Dually we again get a left action of $\mathcal{U}_h(\mathfrak{su}(N))$ on \mathcal{F}_{ahol} and keep the symbol ξ for it. The formula

$$(Y, f) \mapsto w_\lambda^{-1}\xi_Y \cdot (w_\lambda f)$$

defines a left $\mathcal{U}_h(\mathfrak{su}(N))$ – module structure on \mathcal{F}_{ahol} and the representation τ_λ can be identified with the cyclic submodule \mathcal{M}_λ with unit as the cyclic vector and, at

the same time, the lowest weight vector. With the same success we could use the prescription

$$(Y, f) \mapsto (\xi_Y \cdot (fw_\lambda))w_\lambda^{-1}. \quad (5.12)$$

It is not difficult to check that τ_λ can be again identified with a cyclic submodule \mathcal{M}'_λ with respect to this new action and the unit is again the cyclic and the lowest weight vector. From (5.12) follows easily that

$$\begin{aligned} Y \cdot \psi f &= (\xi_Y \cdot (\psi fw_\lambda))w_\lambda^{-1} = (\xi_{Y_{(1)}} \cdot \psi)(\xi_{Y_{(2)}} \cdot (fw_\lambda))w_\lambda^{-1} \\ &= (\xi_{Y_{(1)}} \cdot \psi)(Y_{(2)} \cdot f), \end{aligned}$$

with $\Delta Y = Y_{(1)} \otimes Y_{(2)}$.

6 IRREDUCIBLE REPRESENTATIONS

The left action ξ , dual to the dressing transformation $\mathcal{R} : \mathcal{F}_{ahol} \rightarrow \mathcal{F}_{ahol} \otimes \mathcal{A}_q(SU(N))$, can be expressed explicitly in terms of Chevalley generators. For graphical reasons, we shall write $\xi(Y) \cdot f$ instead of $\xi_Y \cdot f$. Relying on Proposition 5.1, one can pass from the FRT description to Chevalley generators also in the definition of the left module structure on \mathcal{F}_{ahol} . First note that the rules (2.11), (2.7) imply

$$\langle q^{\pm H_j}, A^{-1} \rangle = \langle q^{\pm H_j}, A \rangle^{-1}, \quad \langle X_j^\pm, A^{-1} \rangle = -\langle q^{H_j/2}, A \rangle^{-1} \langle X_j^\pm, A \rangle \langle q^{-H_j/2}, A \rangle^{-1}, \quad (6.1)$$

where A is any invertible square matrix with entries from $\mathcal{A}_q(SU(N))$. In particular (cf. (2.13)),

$$\langle q^{\pm H_j}, U^* \rangle = q^{\mp(E_{jj} - E_{j+1,j+1})}, \quad \langle X_j^+, U^* \rangle = -q^{-1} E_{j,j+1}, \quad \langle X_j^-, U^* \rangle = -q E_{j+1,j}. \quad (6.2)$$

To evaluate ξ one can use the definition (5.1), the formula (5.8) for the dressing transformation and the rules (2.11), (6.1), (6.2):

$$\begin{aligned} \xi(q^{\pm H_j}) \cdot \mathfrak{Z}^{(m)*} &= F^{(m)} q^{\mp(E_{jj} - E_{j+1,j+1})} (E^{(m)} + \mathfrak{Z}^{(m)*}) \\ &\quad \times [F^{(m)} + q^{\mp(E_{jj} - E_{j+1,j+1})} (E^{(m)} + \mathfrak{Z}^{(m)*})]^{-1} \\ &= q^{\mp(E_{jj} - E_{j+1,j+1})} \mathfrak{Z}^{(m)*} q^{\pm(E_{jj} - E_{j+1,j+1})}, \end{aligned} \quad (6.3a)$$

$$\begin{aligned} \xi(X_j^+) \cdot \mathfrak{Z}^{(m)*} &= -q^{-1} F^{(m)} E_{j,j+1} (E^{(m)} + \mathfrak{Z}^{(m)*}) \\ &\quad \times [F^{(m)} + q^{(E_{jj} - E_{j+1,j+1})/2} (E^{(m)} + \mathfrak{Z}^{(m)*})]^{-1} \\ &\quad - F^{(m)} q^{-(E_{jj} - E_{j+1,j+1})/2} (E^{(m)} + \mathfrak{Z}^{(m)*}) \\ &\quad \times [F^{(m)} + q^{-(E_{jj} - E_{j+1,j+1})/2} (E^{(m)} + \mathfrak{Z}^{(m)*})]^{-1} \\ &\quad \times [F^{(m)} - q^{-1} E_{j,j+1} (E^{(m)} + \mathfrak{Z}^{(m)*})] \\ &\quad \times [F^{(m)} + q^{(E_{jj} - E_{j+1,j+1})/2} (E^{(m)} + \mathfrak{Z}^{(m)*})]^{-1} \\ &= -q^{-1} F^{(m)} E_{j,j+1} \mathfrak{Z}^{(m)*} + \mathfrak{Z}^{(m)*} E_{j,j+1} E^{(m)} \\ &\quad + \delta_{jm} q^{-1/2} q^{E_{m+1,m+1}/2} \mathfrak{Z}^{(m)*} E_{m,m+1} \mathfrak{Z}^{(m)*} q^{-E_{mm}/2}, \quad (6.3b) \\ \xi(X_j^-) \cdot \mathfrak{Z}^{(m)*} &= -q F^{(m)} E_{j+1,j} (E^{(m)} + \mathfrak{Z}^{(m)*}) \\ &\quad \times [F^{(m)} + q^{(E_{jj} - E_{j+1,j+1})/2} (E^{(m)} + \mathfrak{Z}^{(m)*})]^{-1} \end{aligned}$$

$$\begin{aligned}
& -F^{(m)} q^{-(E_{jj}-E_{j+1,j+1})/2} (E^{(m)} + \mathfrak{Z}^{(m)*}) \\
& \times [F^{(m)} + q^{-(E_{jj}-E_{j+1,j+1})/2} (E^{(m)} + \mathfrak{Z}^{(m)*})]^{-1} \\
& \times [F^{(m)} - qE_{j+1,j} (E^{(m)} + \mathfrak{Z}^{(m)*})] \\
& \times [F^{(m)} + q^{(E_{jj}-E_{j+1,j+1})/2} (E^{(m)} + \mathfrak{Z}^{(m)*})]^{-1} \\
& = -\delta_{jm} q^{1/2} E_{m+1,m} - qE_{j+1,j} \mathfrak{Z}^{(m)*} + \mathfrak{Z}^{(m)*} E_{j+1,j}. \quad (6.3c)
\end{aligned}$$

All manipulations needed here are quite straightforward. Note, for example, that

$$E^{(m)} [F^{(m)} + q^{\pm(E_{jj}-E_{j+1,j+1})/2} (E^{(m)} + \mathfrak{Z}^{(m)*})]^{-1} = E^{(m)} q^{\mp(E_{jj}-E_{j+1,j+1})/2}.$$

The relations (6.3), expressed directly in terms of entries $z_{st}^{(m)*}$, were already presented in Ref. [6]. Let us recall them ($1 \leq s \leq m < t \leq N$):

$$\begin{aligned}
\xi(q^{\pm H_j}) \cdot z_{st}^{(m)*} &= q^{\pm(\delta_{js}-\delta_{j+1,s}-\delta_{jt}+\delta_{j+1,t})} z_{st}^{(m)*}; \\
\xi(X_j^+) \cdot z_{st}^{(m)*} &= \delta_{j+1,s} z_{jt}^{(m)*}, & \text{for } j < m, \\
&= q^{-(1+\delta_{sm}-\delta_{t,m+1})/2} z_{mt}^{(m)*} z_{s,m+1}^{(m)*}, & j = m, \\
&= -q^{-1} \delta_{jt} z_{s,j+1}^{(m)*}, & j > m; \\
\xi(X_j^-) \cdot z_{st}^{(m)*} &= \delta_{js} z_{j+1,t}^{(m)*}, & \text{for } j < m, \\
&= -q^{1/2} \delta_{sm} \delta_{t,m+1}, & j = m, \\
&= -q \delta_{j+1,t} z_{sj}^{(m)*}, & j > m.
\end{aligned} \quad (6.4)$$

It is quite straightforward to transcribe these relations in terms of generators ζ_{st}^* for $\zeta_{st} = z_{st}^{(s)}$, $1 \leq s < t \leq N$ (cf. (3.20) or (3.21)). Note also that

$$z_{s-1,t}^{(s)} = \zeta_{s-1,t} - \zeta_{s-1,s} \zeta_{st}.$$

The result is given in (6.9) below.

It remains to determine the action on the unit. Recalling that $M = \Lambda^* \Lambda$ and $\Lambda = (\alpha_{jk})$ is upper triangular, one derives immediately from (4.26) that $(\lambda_{N+1} := 0)$

$$\alpha_{st} \cdot 1 = 0 \quad \text{for } s < t, \quad \alpha_{ss}^2 \cdot 1 = \lambda_s, \quad \alpha_{s,s+1}^* \alpha_{ss} \cdot 1 = (\lambda_s - \lambda_{s+1}) z_{s,s+1}^{(s)*}. \quad (6.5)$$

We introduce the substitution

$$\lambda_{j+1}/\lambda_j =: q^{-2\sigma_j}, \quad j = 1, \dots, N-1. \quad (6.6)$$

In virtue of (2.9), the expressions in Chevalley generators follow easily and are given in (6.7) below.

Now we are ready to reformulate Proposition 4.4. Here we employ the quantum coordinate functions on the flag manifold, i.e., the generators ζ_{st} rather than $z_{st}^{(m)}$.

Proposition 6.1. *A left $\mathcal{U}_h(\mathfrak{su}(N))$ - module structure on \mathcal{F}_{ahol} depending on $(N-1)$ scalar parameters $\sigma_1, \dots, \sigma_{N-1}$ is defined unambiguously by the relations $[x] := (q^x - q^{-x})/(q - q^{-1})$*

$$q^{\pm H_j} \cdot 1 = q^{\mp \sigma_j}, \quad X_j^+ \cdot 1 = -q^{-(1+\sigma_j)/2} [\sigma_j] \zeta_{j,j+1}^*, \quad X_j^- \cdot 1 = 0, \quad 1 \leq j \leq N-1, \quad (6.7)$$

and

$$Y \cdot (\zeta_{st}^* f) = (\xi(Y_{(1)}) \cdot \zeta_{st}^*) Y_{(2)} \cdot f, \quad \text{with } \Delta Y = Y_{(1)} \otimes Y_{(2)}, \quad (6.8)$$

for $1 \leq s < t \leq N$, $\forall f \in \mathcal{F}_{ahol}$ and $\forall Y \in \mathcal{U}_h(\mathfrak{su}(N))$. Here ξ is the action dual to the right dressing transformation and it is prescribed on the generators as follows ($1 \leq s < t \leq N$):

$$\begin{aligned} \xi(q^{\pm H_j}) \cdot \zeta_{st}^* &= q^{\pm(\delta_{js} - \delta_{j+1,s} - \delta_{jt} + \delta_{j+1,t})} \zeta_{st}^*, \\ \xi(X_j^+) \cdot \zeta_{st}^* &= \delta_{j,s-1}(\zeta_{s-1,t}^* - \zeta_{st}^* \zeta_{s-1,s}^*) + \delta_{js} q^{-1+\delta_{s+1,t}/2} \zeta_{st}^* \zeta_{s,s+1}^* - q^{-1} \delta_{jt} \zeta_{s,t+1}^*, \\ \xi(X_j^-) \cdot \zeta_{st}^* &= -q^{1/2} \delta_{js} \delta_{s,t-1} - q(1 - \delta_{s,t-1}) \delta_{j,t-1} \zeta_{s,t-1}^* \\ &= -q^{1-\delta_{s+1,t}/2} \delta_{j,t-1} \zeta_{s,t-1}^* \quad (\text{with } \zeta_{ss}^* := 1). \end{aligned} \quad (6.9)$$

The unit is a lowest weight vector for $X_j^- \cdot 1 = 0$, $\forall j$. According to (6.7), the corresponding lowest weight equals

$$\lambda := - \sum_{j=1}^{N-1} \sigma_j \omega_j, \quad (6.10)$$

where ω_j 's are the fundamental weights for $\mathfrak{su}(N)$ defined by $\omega_j(H_k) = \delta_{jk}$. Let \mathcal{M}_λ designate the cyclic submodule of \mathcal{F}_{ahol} with the cyclic vector 1. It is known [30, 31] that \mathcal{M}_λ is determined unambiguously, up to equivalence, by the lowest weight λ and the relation between finite-dimensional irreducible modules and lowest weights is the same in the deformed as well as in the non-deformed case. This observation implies

Proposition 6.2. *The unit in \mathcal{F}_{ahol} is a lowest weight vector corresponding to the lowest weight λ determined by $\lambda(H_j) = -\sigma_j$, $\forall j$. The cyclic submodule $\mathcal{M}_\lambda := \mathcal{U}_h(\mathfrak{su}(N)) \cdot 1$ in \mathcal{F}_{ahol} is a finite-dimensional irreducible $\mathcal{U}_h(\mathfrak{su}(N))$ -module provided all parameters σ_j are non-negative integers.*

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